PRACTICAL MECHANICS

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ELEMENTARY

EXAMPLES IN PRACTICAL MECHANICS

COMPRISING COPIOUS EXPLANATIONS AND PROOFS OF THE

FUNDAMENTAL PROPOSITIONS

BY THE

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PREFACE.

THE following Treatise is designed to be an introduction to the science of *Applied Mechanics;* in this it differs from all the elementary works commonly in use, which are introductory to *Rational Mechanics*. How great a difference is caused by this circumstance will appear from an inspection of the Contents; it may, however, be mentioned that, at the least, one half of the present work has no counterpart in any *Elementary* Treatise that has fallen under the author's notice: that so great a divergence from the usual type should be possible seems sufficient reason for believing that something is wanting in the ordinary works, but how far the present will supply that want is, of course, another question. It was originally intended to be a book of examples, and a supplement to others already in existence; it was, however, soon found that by a few additions it could be made independent, and it was thought that what was gained in point of convenience by completeness, would more than compensate a small increase of size and cost.

The work is intended to comprise two courses; the first is contained in Chapter 1, and in the first sections of Chapters 2, 3, 4, 5, 6, 7, of Part I. and of Chapter 1 of Part II.; the second forms the remainder of the book.

The first course may be read by any one who understands arithmetic, a little algebra, practical geometry, and the rules of mensuration; in many of the examples it is intended that a geometrical construction should take the place of calculation: instances of the use of construction are given in Examples 168, 197, 251, 307, 370, and 377. In this course the principles of the science are simply expounded, their formal demonstration being reserved to the second course; in other words, the order most convenient for teaching and learning has been followed at some sacrifice of the systematic development of the subject. The second course presupposes that the reader is acquainted with Euclid, algebra, and trigonometry, as commonly taught in schools; a very few examples are inserted which require some acquaintance with co-ordinate geometry and the differential calculus*; the reason for their insertion will generally be obvious from the context in which they occur. Frequent use has been made of simple geometrical limits; they will probably present but little difficulty to the reader; he will find some remarks on the subject of limits in Appendix I.

Very many examples require numerical answers; it is hoped that but few of the arithmetical operations will prove laborious to any one who possesses a proper facility in manipulating numbers, and it must be remembered that few things are more important to a learner in the earlier stages of his progress than that he should be continually referred to the numerical results that follow from the formulæ he investigates. Hints and explanations have

* Most of these examples are contained in Chap. 8, Part I.; the others are distinguished by an asterisk.

been freely given in connection with the more difficult examples, and it is hoped they will be found sufficient to enable the reader to complete the solutions, though many of them are important mechanical theorems, and some of them but rarely to be met with (*e.g.* Ex. 134, 149, 297, 365, 469, 509, 515, &c.)

A list is subjoined of the principal works referred to in drawing up the present Treatise; particular instances of obligation are acknowledged in the foot notes in the course of the work. A more explicit recognition of assistance is due to the Rev. H. Moseley, Canon of Bristol: about two hundred of the Examples were given by him to his classes at King's College, London, in the years 1840, 1, 2, 3; these he very kindly placed at the author's disposal, and also gave him permission to use freely his excellent treatise on the "Mechanical Principles of Engineering"—a permission of which great use has been made.

STAFF COLLEGE, August, 1860.

Works referred to.

M. POISSON, Traité de Mécanique.
M. PONCELET, Introduction à la Mécanique Industrielle.
M. MORIN, Aide-Memoire de Mécanique Pratique.
M. MORIN, Notions Fondamentales de Mécanique Pratique.
Dr. T. YOUNG, Lectures on Natural Philosophy.
Rev. W. WHEWELL, D.D., History of the Inductive Sciences.
Rev. H. MOSELEY, Mechanical Principles of Engineering.
Rev. R. WILLIS, Principles of Mechanism.
Dr. RANKINE, Applied Mechanics.

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N.B. — The Chapters and Sections marked with an asterisk (*) form a First Course of Mechanics. (See Preface.)

PRACTICAL MECHANICS.

CHAPTER I.

ON SOME OF THE PHYSICAL PROPERTIES OF MATERIALS.

1. *Properties of materials.*—The object of the present chapter is to serve as an introduction to those that follow. It contains examples illustrative of the more obvious physical properties of the materials commonly used in construction and machinery. These physical properties are (1) Weight; (2) Expansion or Contraction, produced by change of temperature; (3) Elongation and Compression, produced by Pressure; (4) Resistance offered to Rupture by Strain; (5) Resistance offered to Rupture by Compression.

2. *Weight.*—For estimating the weight of masses with sufficient accuracy it may be assumed that the weight of a cubic foot of water is 1000 oz. This number is easily remembered, and is within a very little of the truth. In every example contained in the following pages wherein the weight of masses is concerned, it will be assumed that the weight of a cubic foot of water is 1000 oz., unless the contrary is specified. As a matter of fact, a cubic foot of pure water at 39° F. (when its density is greatest) weighs 998·8 oz. It may also be convenient for the reader to remember that a gallon contains 277·274

d

cubic inches, and that a gallon of water at the standard temperature (62° F.) weighs 10lbs.

Ex. 1.—A reservoir is internally 12ft. long, 5ft. wide, and 3ft. deep: determine the weight of the water it contains when full, and the error produced by considering that each cubic foot weighs 1000 oz.

Ans. Weight, 5 tons, 0cwt. 50lbs.
Error. 13½lbs.

Ex. 2.—A cylindrical boiler terminated by plane ends, is internally 15 ft. long and 4 ft. in diameter; through the lower half pass lengthwise 50 fire tubes, 3 in. in external diameter: determine the volume and weight of the water contained in it when the surface of the water passes through the centres of the ends. *Ans.* Vol. 57·43 cubic ft.
Weight, 1 ton, 12cwts. 0qr. 8·5lbs.

Ex. 3.—The surface of a pond measures 10 acres; in the course of a period of dry weather the surface falls 1½ in. by evaporation: what is the weight of the water that has been withdrawn? *Ans.* 152 tons, nearly.

3. *Specific gravity.*—The specific gravity of a solid or liquid substance means the proportion which the weight of a certain volume of that substance bears to the weight of an equal volume of water; thus when it is stated that the specific gravity of cast iron is 7·2070, it means that a cubic foot, or a cubic inch, &c. of cast iron weighs 7·2070 times as much as a cubic foot, cubic inch, &c. of water; consequently a cubic foot of cast iron will weigh 7207oz., and in general, if S is the specific gravity of a substance, a cubic foot of it will weigh 1000S oz., at least with sufficient accuracy in almost all cases. The following table gives the specific gravities of some common materials:—

TABLE I.

SPECIFIC GRAVITIES.

METALS.

Platinum (laminated)	22·0690	Brass (cast)	8·3958
Pure Gold (hammered)	19·3617	Steel (hard)	7·8163
Gold 22 caret (do.)	17·5894	Iron (cast)	7·2070
Mercury	13·5681	Iron (wrought)	7·7880
Lead (cast)	11·3523	Tin (cast)	7·2914
Pure Silver (hammered)	10·5107	Zinc (cast)	7·1908
Copper (cast)	8·7880		

STONES AND EARTH.

Marble (white Italian)	2·638	Portland Stone	2·145
Slate (Westmoreland)	2·791	Coal (Newcastle)	1·2700
Granite (Aberdeen)	2·625	Brick (Red)	2·168
Paving Stone	2·4158	Clay	1·919
Mill Stone	2·4835	Sand (River)	1·886
Grindstone	2·1429	Chalk (mean)	2·315

WOODS (DRY).

Elm	0·588	Oak (English)	0·934
Fir (Riga)	0·753	Teak (Indian)	0·657
Larch	0·522	Cork	0·240
Mahogany (Spanish)	0·800		

1 foot length of Hempen rope weighs in lbs. 0·045 × (circ. in inches)^2.
1 „ „ Cable weighs in lbs. 0·027 × (circ. in inches)^2.
1 cubic foot of Brickwork weighs 112 lbs.

NOTE.—The above numbers, where printed to four places of decimals, are taken from Dr. Young's Lectures on Natural Philosophy, v. ii. p. 503; where printed to three places of decimals, from Mr. Moseley's Mechanics of Engineering, 1st ed. p. 622. A definite specific gravity is assigned to each substance to prevent ambiguity in working the following examples. It will be remarked, however, that different specimens of the same substance have different specific gravities: thus, of 16 specimens of cast iron the specific gravities have been found to vary from 7·295 to 6·963. The reader must, therefore, bear in mind that the numbers in the text give mean values from which the specific gravity of any specimen of a given substance will not largely vary—though the limits of variation are greater with some substances than with others. A similar remark applies to all quantities determined by experiment.

Ex. 4.—What is the weight of a rectangular block of marble 63 ft. long, and in section 12 ft. square? *Ans.* Weight, 667 tons, 14 cwts. 3 qrs.

Ex. 5.—The girth of a tree is 3 ft. at top, 3 ft. 9 in. at bottom, it is 14 ft. long. Determine its weight according as it is larch, oak, or mahogany. Also, its value at the following prices: larch, 2*s.* 6*d.*; oak, 7*s.*; mahogany, 19*s.* per cubic foot rough.

Ans. Vol. 12·74 cubic ft.

Weight. Larch, 415·6 lbs. Oak, 743·7 lb. Mah. 637 lbs.

Price. „ 1*l.* 11*s.* 10*d.* „ 4*l.* 9*s.* 2*d.* „ 12*l.* 2*s.* 1*d.*

[The volume to be determined as the frustum of a cone.]

Ex. 6.—Find the weight of a rectangular mass of oak, 12 ft. long, 4 ft. broad, and 2½ ft. thick. What would be the weight of a mass of granite of the same dimensions? *Ans.* Oak, 62 cwts. 2 qrs. 5 lbs.
Granite, 175 cwts. 2 qrs. 3½ lbs.

Ex. 7.—Find the separate weights of a cast iron ball, 4 in. in radius, and of a copper cylinder 3 ft. long, the diameter of whose base is 1 in. Determine also the diminution in the weight of the ball if a hole were cut through it which the cylinder would exactly fit, the axis of the cylinder passing through the centre of the sphere. Also, find the error that results from considering the part cut away a perfect cylinder.

Ans. Weight of sphere, 1118·09 oz.
„ cylinder, 143·8 oz.
Weight of part cut from sphere, 26·204 oz.
Error. 0·102 oz.

Ex. 8.—If a 10 in. shell were of cast iron, and were 2 in. thick, what would be its weight supposing it complete? If the weight of a 10 in. shell were 86 lbs. what would be its thickness supposing it complete?

Ans. (1) 107 lbs. (2) 1·41 in.

Ex. 9.—A hammer consists of a rectangular mass of wrought iron, 6 in. long, and 3 in. by 2 in. in section; its handle is of oak, and is a cylinder 3 ft. 6 in. long, on a base 1 in. in radius. Determine its weight.

Ans. 12·73 lbs.

Ex. 10.—A pendulum consists of a cylindrical rod of steel 40 in. long, on a base whose diameter measures $\frac{1}{4}$ in.; to the end of this is screwed a steel cylinder $\frac{1}{2}$ in. thick, and $1\frac{1}{2}$ in. in radius, which fits accurately a hollow cylinder of glass, containing mercury 6 in. deep, the glass vessel weighing 3 oz. Determine the weight of the pendulum. *Ans.* 360·95 oz.

Ex. 11.—Determine the weight of a leaden cone whose height is 1 ft. and radius of base 6 in.; determine also the external radius of that hollow cast iron sphere which is 1 in. thick, and equals the cone in weight.

Ans. (1) 185·74 lbs. (2) 8·02 in.

Ex. 12.—A rectangular mass of cast iron 6 ft. long, 6 in. wide, and 3 in. deep, has fitted square to its end a cube of the same materials whose edge is $1\frac{1}{2}$ ft. long; find its weight. *Ans.* 1858 lbs.

Ex. 13.—It is reckoned that a foot length of iron pipe weighs 64·4 lbs. when the diameter of the bore is 4 in. and the thickness of the metal $1\frac{1}{4}$ in.: what does this assume to be the specific gravity of iron? *Ans.* 7·197.

Ex. 14.—A cast iron column 10 feet high and 6 inches in diameter will safely support a weight of $17\frac{1}{2}$ tons, whether it be solid, or hollow and 1 in. thick; determine :—(1) the weight of a solid column; (2) the number of equally strong hollow columns that can be made out of 500 solid columns; (3) the price of 500 solid columns at 10*s.* per cwt. and of 500 hollow columns at 11*s.* 3*d.* per cwt.; (4) the cost of sending the 500 solid and the 500 columns to a given place at the rate of 10*s.* 6*d.* per ton.

Ans. (1) 884·2 lbs. (2) 900. (3) 1974*l.* 3*s.* solid. 1233*l.* 16*s.* hollow. (4) 103*l.* 13*s.* solid. 57*l.* 12*s.* hollow.

Ex. 15.—Determine the weight of a hollow leaden cylinder whose length is 3 in., internal radius $1\frac{1}{2}$ in., and thickness $1\frac{1}{2}$ in. *Ans.* 26·121 lbs.

Ex. 16.—Determine the weight of a grindstone 4ft. in diameter and 8in. thick, fitted with a wrought iron axis of which the part within the axis is 2 in. square, and the projecting parts each 4 in. long with a section 2 in. in diameter. *Ans.* 1135·7 lbs.

Ex. 17.—Determine the weight of an oak door 7 ft. high, 3 ft. wide and $1\frac{1}{2}$ in. thick. *Ans.* $153\frac{1}{4}$ lbs.

Ex. 18.—There is a fly wheel of cast iron the external radius of whose rim is 5 ft. and internal radius 4 ft. 6 in.; it is 4 in. thick and is connected with the centre by 8 spokes 4 in. wide and 1 in. thick, strengthened by a flange on each side 1 in. square (so that their section is a cross 4 in. long and 3 in. wide) each spoke is 4 ft. long; the centre to which they join the rim has the same thickness as the rim, is solid, and (of course) 6 in. in radius: determine the weight of the whole. *Ans.* 2948·7 lbs.

Ex. 19.—There are 2 rooms each 100 ft. long and 30 ft. wide; the one is floored with oak planking $1\frac{1}{4}$ in. thick; the other with deal planking (Riga fir) $1\frac{1}{2}$ in. thick. Determine the weights of the floors and their cost, the price of deal being 3*s.* and oak 7*s.* per cubic feet.

Ans. Deal floor weighs 17648 lbs. costs 56*l.* 5*s.*
Oak „ 18242 lbs. „ 109*l.* 7*s.* 6*d.*

Ex. 20.—A cubic foot of copper is drawn into wire $\frac{1}{16}$ of an inch in diameter; what length of wire is made? *Ans.* 46936 ft.

Ex. 21.—It is said that gold can be drawn into wire one millionth part of an inch thick; what will be the length of such a wire that can be made from an ounce of pure gold. *Ans.* 1793448 miles.

Ex. 22.—It is said that silver leaf can be made $\frac{1}{150000}$ of an inch thick; how many ounces of silver would be required to make an acre of such silver leaf? *Ans.* 6·875 oz.

4. *Brickwork.*—The measurement and determination of the weight of a mass of brickwork depend upon the following data:—

(1.) A rod of brickwork has a surface of 1 square rod (or $30\frac{1}{4}$ square yards) and a thickness of a brick and a half, *i. e.* of 1ft. $1\frac{1}{2}$in., or it contains 306 cubic feet.

(2.) A rod of brickwork contains about 4500 bricks in mortar, or 5000 bricks laid dry

(3.) A rod of brickwork requires $3\frac{1}{2}$ loads (*i. e.* $3\frac{1}{2}$ cubic yards) of sand and 18 bushels of stone lime.

(4.) A brick measures $8\frac{3}{4} \times 4\frac{1}{4} \times 2\frac{3}{4}$ inches, *i. e* a quarter of an inch each way less than $9 \times 4\frac{1}{2} \times 3$ inches.

(5.) A bricklayer's hod measures $16 \times 9 \times 9$ inches, and can contain 10 or 12 bricks.*

Ex. 23.—How many rods of brickwork are there in a square tower 117 ft. high and 28 ft. by 7 ft. at its base, externally, and 3 bricks thick? Determine the number of bricks required to build the tower and their price at 1*l.* 10*s.* per thousand.

Ans. (1) 52·43 rods. (2) 236,000 bricks. (3) 354*l.*

Ex. 24.—A tower the base of which measures externally 9ft. square is 50ft. high and 2 bricks thick; how many bricks are required to build it, and how many loads of sand and bushels of lime? Determine also the cost of the materials if the bricks cost 1*l.* 10*s.* per thousand, sand 5*s.* 4*d.* per load, and lime 1*s.* 8*d.* per bushel.

Ans. (1) 7·35 rods. (2) 33,000 bricks, $25\frac{3}{4}$ loads of sand, $132\frac{1}{2}$ bushels of lime. (3) Cost 67*l.* 8*s.* 2*d.*

Ex. 25.—How many rods of brickwork are there in a reservoir of a rectangular form, the internal measurements of which are 20ft. long 6ft. wide and 12ft. deep; the work being 2 bricks thick, viz. both walls and floor; and the reservoir being open at the top? *Ans.* 4·43.

Ex. 26.—How many rods of brickwork are there in a wall 360ft. long, 17 ft. high and 2 bricks thick; and determine the cost of the material from the data in Ex. 24. *Ans.* (1) 30 rods. (2) 275*l.* 10*s.*

Ex. 27.—If the wall in the last example had an additional 2ft. of foundation 3 bricks thick, and were supported by 20 square buttresses reaching to the top of the wall 2 bricks thick on foundations 3 bricks thick and measuring $2\frac{1}{2}$ ft. in a direction perpendicular to the face of the wall; determine the number of rods of brickwork in the foundations and buttresses.

Ans. 10·2 rods.

Ex. 28.—What would be the cost of the carriage of the bricks in the wall described in the last two examples at 5*s.* 6*d.* per ton?

Ans. 169*l.* 2*s.* 6*d.*

Ex. 29.—The following are the actual dimensions of the brickwork of the outer shell of the chimney of St. Rollox, Glasgow. Commencing from the top, there are five divisions; the tops of these divisions are respectively $435\frac{1}{2}$, $350\frac{1}{2}$, $210\frac{1}{2}$, $114\frac{1}{2}$, $54\frac{1}{2}$, above the ground; the external diameters at the *tops* of the divisions are respectively 13ft. 6in., 16ft. 9in., 24ft., 30ft. 6in., 35ft. The diameter on the ground is 40ft.; the thickness of the divisions are respectively $1\frac{1}{2}$, 2, $2\frac{1}{2}$, 3, and $3\frac{1}{2}$ bricks; below ground the brickwork reaches 14ft., with a uniform external diameter of 40 ft.; the

* Mr. Weale's Contractor's Handbook. The Handbook states that a hod contains 20 bricks, but this must be a misprint.

first 8 feet are 3 ft. thick; in the remaining 6 feet the thickness gradually increases to 12 ft. thick. Determine the number of rods of brickwork contained in the chimney; the number of thousand bricks employed, their cost at 1*l.* 11*s.* 3*d.* per thousand; also, if the mortar were of sand and stone lime, determine the number of loads of sand and bushels of stone lime required, and their cost at 5*s.* 4*d.* per load, and 1*s.* 8*d.* per bushel, respectively.

[The surface of each division of the chimney may be considered as that of a conic frustum; the real volume of each division will be the difference between the volumes of two conic frustums. A sufficiently close approximation may be obtained by multiplying the mean surface by the thickness and considering the slant side equal to the height; the volume of the part below ground is to be determined accurately.]

Ans. (1) 2188 rods, or 981,000 bricks. (2) Cost of bricks, 1532*l.* 16*s.* 3*d.* (3) 763 loads of sand, costing 203*l.* 9*s.* 4*d.* (4) 3924 bushels of lime, costing 327*l.*

5. *Expansion and contraction by heat.*—It is found that all bodies experience a small change of volume on the application of heat. In general the change is one of increase *, and with sufficient accuracy may be considered to obey the following law within moderate ranges of temperature. If a volume V be increased by k V for an addition of one degree of heat, it will be increased by $n \times k$V for an addition of n degrees of heat, *i. e.* the increase of volume is proportional to the increase of temperature. It must be remembered that the same rule is true of the expansions in *length* which a body experiences from an increase of temperature. In order to fix the conception of a degree of heat it will be proper to mention that when heat is applied to ice the water produced by melting retains a constant temperature until the whole of the ice is melted. This temperature serves as one fixed point, and is called the freezing point. Moreover, boiling water in free contact with the air also keeps at a constant temperature (at least when the barometer stands at a given height). This fact, therefore, supplies a second fixed point, and is called boiling

* Water, near freezing point, is a conspicuous exception.

point, the boiling point being determined when the barometer stands at 30 inches. These two points being fixed, the graduation is then arbitrary. The scale on Farenheit's thermometer (which is commonly used in England) is constructed by dividing the space between the freezing and boiling points into 180 equal parts, termed degrees, and by commencing the graduation 32° below freezing point, so that the freezing point is marked 32°, the boiling point 212°. In the centigrade thermometer (commonly used in France) the graduation begins at the freezing point, and the interval between the freezing and boiling points is divided into 100 equal parts called degrees.* It is easy to see that if at any temperature Fahrenheit's thermometer stood at F° and the centigrade at C°, we should have

$$\frac{F^\circ - 32}{180} = \frac{C^\circ}{100}$$

Ex. 30.—The density of water is greatest at 3°·9 on the centigrade scale: what is the same temperature called on Fahrenheit's scale?

Ans. 39°·02 F.

Ex. 31.—The standard temperature commonly referred to in English experiments is 60° F.; what would the same temperature be called in France? *Ans.* 15°·55 C.

Ex. 32.—If the centigrade thermometer stood at 5° below zero, or at − 5° C, what would the same temperature be marked on Fahrenheit's scale?

Ans. 23° F.

Ex. 33.—What degree on the centigrade scale would be equivalent to − 4° on Fahrenheit's scale? *Ans.* − 20° C.

The following Table gives the fractional part of the whole by which a substance expands when heated:—†

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* In Reaumur's thermometer the freezing point was marked zero, and the boiling point 80°; consequently $\frac{F^\circ - 32}{180} = \frac{R^\circ}{80}$

† From Dr. Young's Natural Philosophy, vol. ii. p. 390.

TABLE II.

Expansion produced by Heat.

	Temperature raised from 32° to 212° F.	Temperature raised 1° F.	Authority.
In length. Glass Tube	0·00077615	0·00000431	Roy.
„ Platinum	0·000856	0·00000476	Borda.
„ Cast Iron	0·0011094	0·00000617	Lavoisier.
„ Wrought Iron	0·001156	0·00000642	Borda.
„ Steel rods	0·0011447	0·00000636	Roy.
„ Brass rods	0·0018928	0·00001052	Roy.
„ Lead . .	0·002867	0·00001592	Smeaton.
„ Copper .	0·001700	0·00000944	Smeaton.
In bulk. Mercury .	. .	0·00010415	Roy.
„ Mercury in glass (apparent)	. .	0·00008696	Committee of Royal Society.

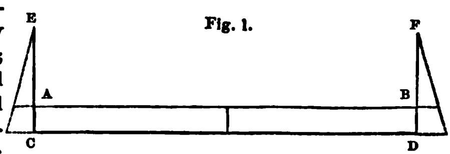
Ex. 24. - The length of the base line of the Ordnance Survey on Hounslow Heath, was found to be 27,404 ft.; this was measured first by glass tubes, and then by steel chains; if, in correcting the glass rods for temperature a uniform error of 1° in excess had been committed, and in correcting the steel chain an error of 1° in defect had been committed, what would have been the difference between the apparent measurements? *Ans.* 3·51 in.

Ex. 35.—If the wrought iron rails on a railway are 10 miles long when at a temperature of 32° below freezing, by how much will they lengthen if their temperature is raised to 88° F. *Ans.* 29·83 ft.

Ex. 36.—Ramsden's brass yard exceeded Shuckburg's by 0·002505 of an inch; what would be the difference of their temperatures when accurately the same length? *Ans.* 6°·6 F.

Ex. 37.— Two rods, respectively of iron and brass, AB and CD are fastened together in the middle; they are accurately the same length, at 62° F.; to their ends are fastened by pivots tongues CAE and DBF which are perpendicular to the bars, at 62° F.; in consequence of the unequal expansion or contraction of the bars the tongues will assume different positions, as shown by the dotted lines; it is required to determine the length of CE, that the point E may remain unmoved by the expansion or contraction of the bar. The length of AB is 10 ft. and the distance AC is 1·725 in. *Ans.* CE = 4·426 in.

Fig. 1.



Ex. 38.—If the expansion in length of a substance is *e* times the length at

given temperature, show that the expansion in volume will be very nearly 3 e times the volume at that temperature.

Ex. 39.—The volume of a mass of lead being a cubic foot at 60° F. what will be its volume at 0° F.? and what at 88° F.?

Ans. At 0° F. 0·9971344 cubic ft.
At 88° F. 1·00133728 cubic ft.

Ex. 40.—There is half a cubic inch of mercury in a thermometer at 32° F.; when the temperature is raised to 92° F. the mercury ascends 4 in.; what is the diameter of the bore of the glass tube?

Ans. 0·0288 in.

6. *Elongation produced by strain.*—The principle on which this determination is made is the following:—Suppose the length of a beam or bar to be L feet, the area of its section to be K square inches, then if by the application of a strain of P lbs. its length becomes $L + l$, it appears from experiment that

$$l : L :: \frac{P}{K} : E$$

where E is a constant number depending on the nature of the material, and is called the Modulus of Elasticity.

It is found that all substances obey this law when the degree of extension does not transgress certain limits; the limits are different in different substances, and in many are very narrow. It appears also that within these limits (*i. e.* the limits of elasticity) a strain producing a certain degree of extension will, if applied in the opposite direction so as to become a thrust, produce an equal degree of compression.

It will be observed that $\frac{P}{K}$ is the strain or thrust per square inch on the section of the beam or bar. It is also plain that if $\frac{P}{K}$ were equal to E then would l be equal to L, so that the modulus of elasticity is that strain per square inch of the section of a bar which would double its length if its elasticity continued perfect. It is, perhaps, unnecessary to remark that scarcely any substance has limits of elasticity any way approaching this in extent.

TABLE III.

Moduli of Elasticity.*

Material.	Modulus.	Material.	Modulus.
Wrought iron bars	29,000,000	Oak (English) .	1,450,000
Cast iron . .	17,000,000	Larch . .	1,050,000
Cast brass . .	8,930,000	Fir (Riga) .	1,330,000
Steel (hard) .	29,000,000	Elm . . .	700,000
Copper wire .	17,000,000		

Ex. 41.—By how much would a bar of wrought iron $\frac{1}{4}$ of an inch square and 100ft. long lengthen under a strain of 2 tons (neglecting the weight of the bar)? *Ans.* 0·247 ft.

Ex. 42.—Determine the elongation of a steel bar 2 in. square and 40 ft. long when subjected to a strain of 40 tons. What would have been its elongation had it been of cast brass? *Ans.* Steel 0·03 ft. Brass 0·1 ft.

Ex. 43.—A bar of wrought iron 2 in. square has its ends fixed between two immoveable blocks when the temperature is 20° F.; what pressure will it exert against them if the temperature becomes 96° F. *Ans.* $25\frac{1}{4}$ tons.

Ex. 44.—A wall of brickwork 2 ft. thick and 12 ft. high is supported by columns of oak 6 inches in radius 18 ft. high and 14 ft. apart from centre to centre; determine the thrust per square inch exerted on the section of the columns, and the amount of their compression.

Ans. (1) 332·7lb. (2) $\frac{1}{20}$ in. nearly.

Ex. 45.—In the last example if the wall had been of Portland stone and $1\frac{1}{4}$ft. thick, what would have been the pressure per square inch, and the degree of compression. *Ans.* (1) 248·9 lb. (2) $\frac{3}{80}$ in.

Ex. 46.—In the last example if the oak column were replaced by a wrought iron bar 2 inches square, what would be the degree of compression? and at what temperature would the iron rod have the same length as it has when unpressed at 32° F.? *Ans.* (1) $\frac{21}{400}$ in. (2) 69·8° F.

Ex. 47.—A bar of wrought iron a square inch in section is fixed firmly between two *immoveable* blocks which are 50 ft. apart; if the temperature is raised 50° F. above that which the bar had when fixed find the pressure produced against these blocks. *Ans.* 9309 lbs.

Ex. 48.—In the last example, if only one of the blocks were immoveable and the other were capable of revolving round a joint 12 ft. below the point, at which it is met by the rod, determine the angle through which it will be turned by the expansion of the rod. *Ans.* 0°·4′ 36″.

Ex. 49.—It is observed that the two opposite walls of an ancient building

* Based on Mr. Moseley's Mech. Eng. p. 622 compared with Mr. Rankine's Applied Mechanics, p. 631.

are each 3° out of the vertical, the inclination being outward; to bring them into the perpendicular the following means are employed: at certain intervals iron bars are placed across the building, their ends passing through the walls and projecting on the outside, on these ends strong plates or washers are screwed; the rods are then heated and expand, in this state the washers are screwed tightly against the outside of the walls and the rods allowed to cool, when they contract and draw the walls together; the process being continued until the walls become vertical.* If we suppose the rods to be 50 ft. long and 3 square inches in section, and to be fastened 15 ft. above the joint of the masonry, round which walls will be made to turn; and if the range of temperature is from 60° F. to 240° F.; determine the number of times the bars must be heated before the operation is complete, and the pressure with which the walls would be drawn together if they were entirely immoveable.

Ans. (1) 27 times. (2) 100,572 lbs.

7. *Resistance to rupture by tearing or tenacity.*—When a strain which elongates a bar attains a certain magnitude, the bar will break. If we determine by experiment the force in lbs. per square inch, we obtain the *tenacity* of the substance. It is manifest that the strain which will tear a bar whose section is n square inches will be n times the tenacity.

TABLE IV.

TENACITIES.

Material.	Tenacity.	Material.	Tenacity.
Wrought Iron (bars) .	67,200 lbs.	Oak (English .	17,300 lbs.
CastIron (average)	16,500 „	Larch . .	10,000 „
Iron wire ropes	90,000 „	Fir (Riga) .	12,000 „
Cast Brass .	18,000 „	Elm . . .	13,500 „
Copper wire .	60,000 „	Hempen ropes .	5,600 „

Ex. 50.—How great a strain will a cylindrical bar of wrought iron bear which is $\frac{1}{4}$ of an inch in diameter? and by what fraction of its length would it elongate under this strain if the elasticity continued perfect?

Ans. (1) 3298·67 lbs. (2) 0·0023.

Ex. 51.—How many iron wires $\frac{1}{10}$ of an inch in diameter must be put together to sustain a strain of 3 tons? *Ans.* 12.

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* The walls of Armagh Cathedral were restored by this process. Daniell's Chemistry, p. 103.

Ex. 52.—What is the length of a bar of wrought iron which being suspended vertically would break by its own weight? *Ans.* 19,880 ft.

Ex. 53.—What strain will a bar of oak $1\frac{1}{2}$ in. square sustain? *Ans.* 38,925 lbs.

Ex. 54.—What strain will a cylindrical bar of larch $1\frac{1}{2}$ in. in diameter sustain? *Ans.* 17,671 lbs.

Ex. 55.—If a rope be made of wires whose diameter is d, show that the number of wires in each square inch of the section of the rope is very nearly given by the formula $\frac{2}{\sqrt{3}\, d^2}$ or $\frac{8}{7\, d^2}$

Ex. 56.—How many wires $\frac{1}{10}$ of an inch in diameter must be put together to form a rope a square inch in section. *Ans.* 115.

Ex. 57.—What is the difference in the number of wires $\frac{1}{20}$ of an inch in diameter which would form a rope with a section of a square inch as determined by the formulæ in Ex. 55? *Ans.* 4·8.

Ex. 58.—Show that the number of lbs. weight in a foot length of iron wire is given by the formula (circ. in inches)^2 × 0 244 very nearly; the specific gravity of iron wire being assumed to be the same as that of wrought iron.

Ex. 59.—Show that if a rope of hemp has the same strength as another of iron wire; the circumference of the latter is about $\frac{1}{4}$, and its weight about $\frac{1}{8}$ of the former.

8. *Resistance to rupture by compression.*—There are as many as five forms which the results of crushing assume in different bodies. They are enumerated as follows by Mr. Rankine *: —

(1.) *Crushing by splitting*, when the substance divides in a direction nearly parallel to the direction of the pressure. This occurs in the case of hard homogenous substances of a glassy texture.

(2.) *Crushing by shearing*, when the substance divides along a plane inclined at a certain angle to the direction of the force, the upper part of the substance sliding upon the lower. This fact was ascertained, and its conditions investigated, by Mr. Hodgkinson. It takes place in the case of substances of a granular texture, such as cast iron, and most

* Applied Mechanics, p. 303. See also Mr. Moseley's Mechanics of Engineering, pp. 549, 579.

kinds of stone and brick. To exhibit its effects the block to be crushed must be at the least one half higher than it is thick. In the above cases the resistance to crushing is considerably greater than the tenacity. In the case of cast iron the resistance is more than *six* times the tenacity.

(3.) *Crushing by bulging*, when the material spreads like compressed dough. This takes place with ductile substances, such as wrought iron in short blocks. In this case the resistance is somewhat less than the tenacity, being with wrought iron about $\frac{2}{3}$ of the tenacity.

(4.) *Crushing by crippling*, which is characteristic of fibrous substances, and takes place when the thrust acts along the fibres in timbers and in bars of wrought iron that are too long to yield by bulging. It consists in a lateral yielding, and sometimes separation of the fibres. In the case of dry timber the resistance is about $\frac{1}{2}$ of the tenacity, in the case of moist timber about $\frac{1}{4}$th of the tenacity; consequently, moist timber is only half as strong as dry when subjected to a crushing force.

(5.) *Crushing by crossbreaking*, which is the mode of fracture in columns and struts where the length greatly exceeds the diameter. Under the breaking load they yield sideways, and are broken across like beams under a transverse pressure.

TABLE V.

CRUSHING PRESSURE IN LBS. PER SQUARE INCH.

Material.	Pressure.	Material.	Pressure.
Wrought Iron .	36,000	Granite (average)	8,000
CastIron (average)	112,000	Oak (English) dry	9,500
Cast Brass .	10,300	Larch dry .	5,500
Brick . .	800	Fir (Riga) dry .	6,000
Sandstone .	4,000	Elm . .	10,300
Limestone (granular) . .	4,000		

Ex. 60.—What must be the height of a column of cast iron producing that pressure per square inch which would crush a short column of the same material? *Ans.* 35,805 ft.

Ex. 61.—Compare the heights of columns of cast iron, wrought iron, cast brass, and larch fir, which would produce the pressure per square inch requisite for crushing short columns of their respective materials?

Ans. 1·475 : 0·439 : 0·116 : 1.

9. *Ultimate and proof strength and working stress.*—It must be borne in mind that no material is in practice subjected to the strain or thrust which it is capable of supporting. This will appear very clearly from the following definitions *: —

(1.) *The ultimate strength* of a solid is the stress required to produce fracture in some specified way.

(2.) *The proof strength* is the stress required to produce the greatest strain in some specified way consistent with safety. A stress exceeding the proof strength, though it does not produce immediate fracture, will produce it by long application or frequent repetition.

(3.) *The working stress* is always made less than the proof *strength* in a certain ratio determined by experience.

In the cases of wrought-iron boilers, timber, brick, and stone, the *ultimate strength* is from 2 to 3 times more than the *proof strength,* and from 8 to 10 times the *working stress.* In the following examples the *working stress* is assumed to be $\frac{1}{10}$th of the ultimate strength:—

Ex. 62.—A wall of brickwork 3 ft. thick, is supported at intervals of 10 ft. by sandstone columns 9 in. in diameter; to what height can the wall be carried? *Ans.* 7·6 ft.

Ex. 63.—If in the last example the columns had been of brickwork 2 ft. thick, to what height would the work then be carried? *Ans.* 10·8 ft.

Ex. 64.—To what height could the wall in Ex. 44 be carried with safety so far as the strength of the columns is concerned? *Ans.* 34·26 ft.

Ex. 65.—Make the same determination with regard to Ex. 45.

Ans. 45·8 ft.

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* Rankine, Applied Mechanics, p. 273.

Ex. 66.—What would have been the heights in each of the last examples if the columns had been of brickwork? What if of limestone? What if of granite? *Ans.* **Brickwork, 2·9 ft. 3·9 ft.**
Limestone, 14·4 ft. 19·3 ft.
Granite, 28·9 ft. 38·6 ft.

Ex. 67.—A wall of brickwork, 50 ft. high and 3 ft. thick, is to be carried by columns of brick 20 ft. apart, from centre to centre; determine the least diameter consistent with safety. Make the same determination if the columns were of granite. *Ans.* **$73\frac{1}{8}$ in. brickwork. $23\frac{1}{8}$ in. granite.**

10. *Strength of cast iron columns.*—The columns in the preceding examples are supposed to follow the law of the crushing of short columns. It may be instructive to add the following particulars, which have reference to the crushing of cast iron columns exceeding that length. The greatest part of our knowledge of this subject is due to experiments conducted by Mr. Hodgkinson, who thus states his conclusions with regard to the form of the ends of iron columns:—"1st. A long circular pillar, with its ends flat, is about three times as strong as a pillar of the same length and diameter with its ends rounded in such a manner that the pressure could pass through the axis. . . . . 2nd. If a pillar of the same length and diameter as the preceding has one end rounded and one flat, the strength will be twice as great as that of one with both ends rounded. 3rd. If, therefore, three pillars be taken, differing only in the forms of their ends, the first having both ends rounded, the second having one end rounded and one flat, and the third both ends flat, the strength of these pillars will be as 1—2—3 nearly." Mr. Hodgkinson further considers that the breaking weight w of a hollow column is given in tons by the formula,

$$w = \mathrm{M} \times \frac{\mathrm{D}^{3\cdot5} - d^{3\cdot5}}{l^{1\cdot63}}$$

and that of a solid column by the formula,

$$w = \mathrm{m} \times \frac{\mathrm{D}^{3\cdot5}}{l^{1\cdot63}}$$

where M and m are constants depending on the nature of the iron, D the external and d the internal diameters of the column in inches, and l the length in feet. The values of M and m vary considerably with different kinds of iron, but may be taken at 42 tons. The limits of variation in the values of m are 49·94 and 33·60.*

Ex. 68.—Determine the breaking weight of a solid cast iron column 20 ft. high and 6 in. in diameter. *Ans.* 168·3 tons.

Ex. 69.—Determine the breaking weight of the column in the last example if it were hollow and 1 in. thick. *Ans.* 127·6 tons.

Ex. 70.—Determine the thickness of a column 20 ft. high and 7 in. in external diameter, which is as strong as that in Ex. 68. *Ans.* 0·774 in.

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* Proceedings of the Royal Society, v. viii. p. 318.

CHAP. II.

ON WORK; OR THE EFFICIENCY OF AGENTS.

11. *Definition of work.*—An agent is said to do work when it causes the point of application of the pressure it exerts to move through a certain space; thus a carpenter employed in planing wood *works*, since he causes the point of application of the pressure he exerts to move through a certain space, and the same is true of any agent that works in the sense here intended. For the sake of distinctness it may be observed that the union of *pressure* and *motion* is essential to the conception of *work*; thus when the expansive force of steam lifts the piston of a steam engine it does work. In the boiler, though it produces an enormous pressure on the surface, it does no work, since the pressure is unaccompanied by motion. The unit by which the work of different agents is expressed numerically is called the unit of work; according to the practice of English writers it is defined as follows:—

Def.—The work done when a pressure of 1 lb. is exerted through 1 ft. is called a unit of work.

The following important principle is deducible from this definition. When a pressure of P lbs. is exerted through a space of S ft., it does PS units of work, the pressure being exerted along the line in which its point of application is made to move. For since a unit of work is done when a pressure of 1 lb. is exerted through 1 ft., there must be 2 units of work done when a pressure of 2 lbs. is exerted through 1 ft., 3 units of work when a pressure of 3 lbs. is exerted through 1 ft., and generally P units of work when a pressure of P lbs. is exerted through 1 ft.

Again, since P units of work are done when a pressure of P lbs. is exerted through 1 ft., there must be 2P done when it is exerted through 2 ft., 3P when it is exerted through 3 ft., and generally PS units must be done when the pressure of P lbs. is exerted through S ft.

Ex. 71.—How many units of work are expended in raising 2 cwts. through 30 fathoms? *Ans.* 40,320.

Ex. 72.—The mean pressure on the piston of a steam engine is 15 lbs. per sq. in., the length of the stroke is 6 ft.; if the area of the piston is 448 sq. in., how many units of work are done per stroke? *Ans.* 40,320.

12. *Comparison of the efficiency of agents.*—If the above examples are compared, it will be seen that the work done during each stroke by the steam on the piston of the engine is equivalent to the work expended in raising 2 cwts. through a height of 30 fathoms; and whatever agent raises this weight, it must do as much work as that done by the steam. In these examples we have not considered the *time* in which the work is done; let us then suppose that the engine in Ex. 72 makes 10 strokes per minute; the expansive force of the steam will then do 403,200 units of work per minute. Now, if we suppose an agent, or a number of agents, to raise a weight of 1 ton through 30 fathoms in one minute, they will do exactly 2240×180 or 403,200 units of work per minute. It is plain that under these circumstances the comparison is complete between the efficiency of the expansive force of the steam and the efficiency of the other agents, and that they are reciprocally equivalent. Hence we infer the general principle:—

The number of units of work yielded by any agent in a given time is the true measure of its efficiency or working power.

Of course it follows from this principle that the working powers of two agents are in the ratio of the number of units of work done by them in the same time.

The most familiar instance of this mode of measuring the

power of an agent is furnished by the steam engine, whose efficiency is estimated in horse-power, as when we speak of an engine of "twenty horse-power." From some experiments, Mr. Watt concluded that a horse is capable of yielding 33,000 units of work per minute. The conclusion, as far as regards the efficiency of the animal, is not very correct; it has, however, fixed the meaning of the term horse-power when applied to a steam engine. Hence

Def.—A steam engine works with one horse-power when it yields 33,000 *units of work per minute.*

Of course an engine of n horse-powers yields n times 33,000 units of work per minute.

Ex. 73.—The piston of a steam engine is 15 in. in diameter, its stroke is $2\frac{1}{2}$ ft. long; it makes 40 strokes per minute; the mean pressure of the steam on it is 15 lbs. per square inch; what number of units of work is done by the steam per minute and what is the horse-power of the engine?

***Ans.* 265072 units of work. 8·03 H.-P.**

Ex. 74.—A weight of $1\frac{1}{2}$ tons is to be raised from a depth of 50 fathoms in 1 minute; determine the horse-power of the engine capable of doing the work. *Ans.* $30\frac{6}{11}$ H.-P.

Ex. 75.—The resistance to the motion of a certain weight is 440 lbs. how many units of work must be expended in making this weight move over 30 miles in 1 hour? What must be the horse-power of an engine that does the same number of units of work in the same time?

***Ans.* 69,696,000 units of work. $35\frac{1}{5}$ H.-P.**

13. *Application of the foregoing principles.*—A considerable number of practical questions can be answered by means of the principles already laid down, viz. such questions as the horse-power of the engine required to do a certain amount of work, the time in which an engine of a certain power will do a certain amount of work, &c. . . . . They are all done by following the same method, viz. First, from a consideration of the work to be done, obtain the number of units of work that must be expended in a certain time. Next, from a consideration of the power of the agent obtain the number of units yielded in the same time. One

of these expressions will contain an unknown quantity; but, since by the terms of the question they are equal, they will form an equation from which the unknown quantity can be readily determined.

Ex. 76.—An engine is required to raise a weight of 13 cwts. from a depth of 140 fathoms in 3 minutes; determine its horse-power?

Let x be the required horse-power; then the units of work yielded in 3 minutes will equal $33000 \times x \times 3$; also the number of units of work required to raise 13 cwts. from a depth of 140 fath. equals $13 \times 112 \times 140 \times 6$. And since these two numbers are equal we have

$$33000 \times 3 \times x = 13 \times 112 \times 140 \times 6.$$

$$\therefore \quad x = 12{\cdot}35 \text{ H.-P.}$$

Ex. 77.—In how many minutes would an engine working at 25 horse power raise a load of 12 cwts. from a depth of 160 fathoms?

Ans. 1·564 min.

Ex. 78.—A locomotive engine draws a gross load of 60 tons at the rate of 20 miles an hour; the resistances are at the rate of 8 lbs. per ton; what must be the horse-power of the engine?

[The reader must bear in mind that the work to be done is to overcome a resistance of 480 lbs. through 20 miles in one hour.]

Ans. 25·6 H.-P.

Ex. 79.—What must be the horse power of an engine that raises 20 cubic feet of water per minute from a depth of 200 fathoms?

Ans. $45\frac{5}{11}$ H. P.

Ex. 80.—How many cubic feet of water would an engine working at 100 horse-power raise per minute from a depth of 25 fathoms? *Ans.* 352.

Ex. 81.—How many cubic feet of water will an engine of 250 horse-power raise per minute from a depth of 200 fathoms? *Ans.* 110 cub. ft.

Ex. 82.—It being required to raise 100 cubic feet of water per minute from a depth of 495 ft., what must be the horse-power of the engine?

Ans. $93\frac{3}{4}$ H-.P.

Ex. 83.—There is a mine with three shafts which are respectively 300, 450, and 500 ft. deep: it is required to raise from the first 80, from the second 60, from the third 40 cubic feet of water per minute; what must be the horse-power of the engine? *Ans.* $134\frac{31}{66}$ H.-P.

Ex. 84.—At what rate per hour will a locomotive engine of 30 horse-power draw a train weighing 90 tons gross, the resistances being 8 lbs. per ton? *Ans.* 15·625 miles.

Ex. 85.—What is the gross weight of a train which an engine of 25 horse-power will draw at the rate of 25 miles an hour, resistances being 8 lbs. per ton? *Ans.* 46·875 tons.

Ex. 86.—A train whose gross weight is 80 tons travels at the rate of 20 miles an hour, if the resistance is 8 lbs. per ton what is the horse-power of the engine? *Ans.* $34\frac{2}{15}$ H.-P.

Ex. 87.—An engine working with the same power as that in the last example draws a train at the rate of 30 miles an hour; the resistances being 7lbs. per ton what is the gross weight of the train?

Ans. $60\frac{20}{21}$ tons.

Ex. 88.—What must be the length of the stroke of the piston of an engine, the surface of which is 1500 square inches, which makes 20 strokes per minute, so that with a mean pressure of 12 lbs. on each square inch of the piston, the engine may be of 80 horse-power? *Ans.* $7\frac{1}{3}$ ft.

Ex. 89.—The diameter of the piston of an engine is 80 inches, the length of the stroke is 10 feet, it makes 11 strokes per minute, and the mean pressure of the steam on the piston is 12 lbs. per square inch: what is the horse-power? *Ans.* 201·06 H.-P.

Ex. 90.—Find the horse-power of an engine that will raise in one minute 100 cubic feet of water from a depth of 600 feet? *Ans.* $113\frac{7}{11}$ H.-P.

Ex. 91.—A train weighing 50 tons is to be drawn along a railway at the rate of 20 miles an hour; the resistances being 8 lbs. per ton, find the horse power of the engine. *Ans.* $21\frac{1}{3}$ H.-P.

Ex. 92.—The cylinder of a steam engine has an internal diameter of 3 feet; the length of the stroke is 6 feet; it makes 6 strokes per minute; under what effective pressure per square inch would it have to work in order that 75 horse-power may be done on the piston? *Ans.* 67·54 lbs.

Ex. 93.—What must be the horse-power of a stationary engine that draws a weight of 150 tons along a horizontal road at the rate of 30 miles per hour; friction being 8 lbs. per ton? *Ans.* 96 H.-P.

14. *Modulus of a machine.*—An agent rarely, if ever, does a considerable amount of useful work *directly*, but nearly always through the intervention of a machine, by which the motive power of the agent is so applied as to overcome the resistance in the most convenient manner. For instance, when a steam engine raises water out of a shaft, the motive power is the expansive pressure of the steam on the piston, the resistance to be overcome is the weight of the water, the beam, crank, &c. of the engine are the means by which the motive power is applied so as to overcome the resistance. Now it will be remarked that each part of the machine presents more or less resistance to the motion, so that a certain part of the work done by the motive power must be expended in overcoming these resistances, *i. e.* in reference to the purpose of the machine must be expended

uselessly. The remainder of the work done by the motive power will be expended *usefully* in accomplishing that purpose.

It admits of proof in the case of a machine moving uniformly, that if the number of units of work done by the agent is represented by U, the number expended in overcoming prejudicial resistances by U_0, and the number expended usefully by U_1, all in the same given time, then

$$U = U + U_1$$

It also appears that in most machines U_1 bears to U a constant ratio, so that

$$U_1 = KU$$

where the letter K denotes some proper fraction, depending on the nature of the machine, this fraction is called the modulus of the machine; the following table, taken from General Morin's *Aide-Mémoire de Mécanique Pratique*, gives the value of K for different classes of steam engines:—

TABLE VI.

MODULI OF STEAM ENGINES.

Description of Machine.	Horse-Power.	Value of K. Best Working.	Ordinary do.
Watt's low-pressure engine	4 to 8	0·50	0·42
	10 „ 20	0·56	0·47
	30 „ 100	0·60	0·54
Cornish engines, working by expansion and condensation	up to 30	0·44	0·35
	30 „ 40	0·49	0·39
	40 „ 50	0·57	0·46
	50 „ 60	0·62	0·50
	60 „ 70	0·66	0·53
	70 „ 80	0·82	0·66
	80 „ 100	0·70	0·59
High-pressure engines, working without expansion or condensation	up to 10	0·50	0·40
	10 „ 20	0·55	0·44
	20 „ 30	0·60	0·48
	30 „ 40	0·65	0·52
	above 40	0·70	0·56

Ex. 94.—The diameter of the piston of a steam engine is 60 in.; it makes 11 strokes per minute; the length of each stroke is 8 ft.; the mean pressure per square inch, 15 lbs. The modulus of the engine being 0·65, determine the number of cubic feet of water it will raise per hour from a depth of 50 fathoms.

[The number of units of work done by steam on piston in one hour equals $\pi \times 30^2 \times 8 \times 15 \times 11 \times 60$; this number multiplied by 0·65 will give the number of units usefully spent in raising water; hence the number of cubic feet of water is found.] *Ans.* 7762·87 cub. ft.

Ex. 95.—The diameter of the piston of an engine is 80 in., the mean pressure of the steam is 12 lbs. per square inch, the length of the stroke is 10 ft., the number of strokes made per minute is 11. How many cubic feet of water will it raise per minute from a depth of 250 fathoms, its modulus being 0.6? *Ans.* 42·46 cub. ft.

Ex. 96.—If the engine in the last example had raised 55 cubic feet of water per minute from a depth of 250 fathoms, what would have been its modulus? *Ans.* 0·7771.

Ex. 97.—How many strokes per minute must the engine in Ex. 95 make to raise 15 cubic feet of water per minute from the given depth? *Ans.* 4.

Ex. 98.—What must be the length of the stroke of an engine whose modulus is 0·65, and whose other dimensions and conditions of working are the same as in Ex. 95, if they both do the same useful work?

Ans. 9·23 ft.

Ex. 99.—The diameter of the cylinder of an engine is 80 inches, the piston makes per minute 8 strokes of $10\frac{1}{2}$ ft. under a mean pressure of 15 lbs. per square inch; the modulus of the engine is 0·55. How many many cubic feet of water will it raise from a depth of 112 ft. in one minute?

Ans. 485·78 cub. ft.

Ex. 100.—If in the last example the engine raised a weight of 66,433 lbs. through 90 ft. in one minute, what must be the mean pressure per square inch on the piston? *Ans* 26·37 lbs.

Ex. 101.—If the diameter of the piston of the engine in Ex. 99 had been 85 in. what addition—in horse-power— would that make in the *useful* power of the engine? *Ans.* 13·28. H.-P.

15. *Work of water wheels.*—Hitherto we have considered only one kind of motive power, viz. the pressure of steam. The same principles are applicable to machines worked by any other motive power, as by the muscular force of animal agents, the pressure of moving air, or of falling water. The last of these, viz. the power of falling water, is, next to steam, the most conspicuous example of work done on a large scale by an inanimate agent. We shall therefore

consider somewhat particularly the application of this power in the instance of water wheels.

It is plain that 1 lb. of water, in descending through 1 foot, must accumulate as much work as would be required to raise it through one foot, and hence if P lbs. of water descend through h feet, they will accumulate Ph units of work; and if, moreover, we suppose this water to descend against an obstacle, such as the float boards of a water wheel, the amount of work so accumulated will be done upon the wheel, and this work may then be applied to any useful purpose after a certain deduction has been made on account of prejudicial resistances.

It must be borne in mind that the height of the fall is the difference between the levels of the surface of the water in the reservoir and in the exit canal; in the case of overshot wheels it is supposed that the extreme circumference of the wheel is just in contact with the surface of the water in the exit canal. The height is represented by A B in the accompanying figures; of which fig. 2 represents the

Fig. 2.

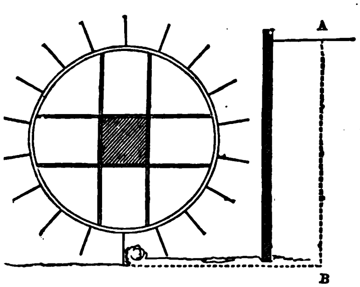
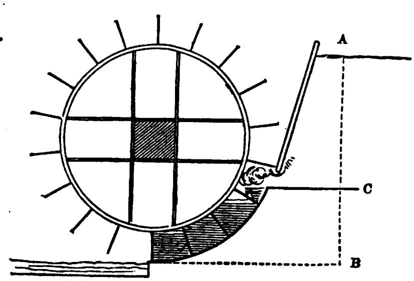
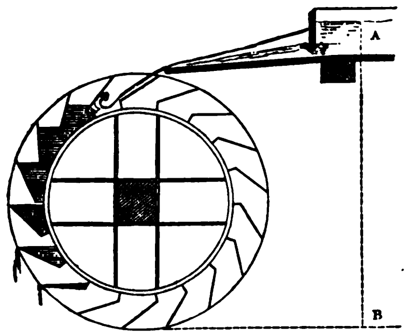


Fig. 3.



ordinary undershot wheel with plane float boards; fig. 3 the breast wheel, in which the water acts upon the float boards considerably above the level of the exit canal. Fig. 4 represents the overshot wheel.

Fig. 4.



The following table exhibits the moduli of various kinds of water wheels. It is founded on results given in General Morin's *Aide-Mémoire.* In the table H denotes the length of the line AB in figs. 2, 3, 4, and h denotes the length of BC in fig. 3: —

TABLE VII.

MODULI OF WATER WHEELS.

Description.	Modulus.
(1) Undershot wheels with flat float-boards . .	0·25 to 0·30
(2) Breast wheels with flat float boards	
(a) when $\frac{h}{H} = \frac{1}{4}$	0·40 to 0·45
(b) " $\frac{h}{H} = \frac{3}{8}$	0·42 " 0·49
(c) " $\frac{h}{H} = \frac{2}{3}$	0·47
(d) " $\frac{h}{H} = \frac{3}{4}$	0·55
(e) " $\frac{h}{H} = 1$	0·65 " 0·70
(3) Breast wheels with curved float boards (Poncelet's construction) . . . . . for H greater than $6\frac{1}{2}$ feet.	0·60 to 0·65
(4) Overshot wheels, when the velocity is small and the buckets half filled	0·70 to 0·75

Ex. 102.—The mean section of a stream is 5 ft. by 2 ft.; its mean velocity is 35 ft. per minute; there is a fall of 13 ft. on this stream, at which is erected a water wheel whose modulus is 0·65; determine the horse-power of the wheel. *Ans.* 5·6. H.-P.

Ex. 103.—In how many hours would the wheel in the last example grind 1000 quarters of wheat, it being assumed that each horse-power will grind one bushel per hour? *Ans.* 1428 hours.

Ex. 104.—How many quarters of wheat will the same wheel grind in 72 hours? *Ans.* 50·41 quarters.

Ex. 105.—Suppose the wheel in Ex. 102 to have replaced an undershot wheel with flat float boards, whose modulus was 0·25, determine the number of quarters of wheat each wheel will grind in 24 hours. *Ans.* (1) 6·5. (2) 16·8.

Ex. 106.—How many cubic feet of water must be made to descend the fall per minute in Ex. 102, 3, that the wheel may grind at the rate of $3\frac{1}{2}$ quarters per hour? *Ans.* 1749·5.

Ex. 107.—Given the stream in Ex. 102, 3, what must be the height of the fall to grind $1\frac{1}{4}$ quarters per hour; first, if the modulus of the wheel is 0·40, next, if it is 0·47, and lastly, if it is 0·65.

Ans. (1) 37·76. (2) 32 ft. (3) 23·2 ft.

Ex. 108.—The mean section of a stream is 8 ft. by 1 ft.; its mean velocity is 40 ft. per minute; it has a fall of $17\frac{1}{2}$ ft.; it is required to raise water to a height of 300 ft. by means of a water wheel whose modulus is 0·7; how many cubic feet will it raise per minute? *Ans.* 13·07 cub. ft.

Ex. 109.—To what height would the wheel in the last example raise $2\frac{1}{4}$ cubic feet of water per minute? *Ans.* $1742\frac{2}{9}$ ft.

Ex. 110.—The mean section of a stream is $1\frac{1}{2}$ ft. by 11 ft.: its mean velocity is $2\frac{1}{2}$ miles per hour; there is on it a fall of 6 ft. on which is erected a wheel whose modulus is 0·7; this wheel is employed to raise the hammers of a forge, each of which weighs 2 tons, and has a lift of $1\frac{1}{2}$ ft.; how many lifts of a hammer will the wheel yield per minute? *Ans.* 142 nearly.

Ex. 111.—In the last example determine the mean depth of the stream if the wheel yields 135 lifts per minute? *Ans.* 1·43 ft.

Ex. 112.—In Ex. 110 how many cubic feet of water must descend the fall per minute to yield 97 lifts of the hammer per minute? *Ans.* 2483 cub. ft.

Ex 113.—Determine how many quarters of corn the mill in Ex. 110 might be made to grind in 6 days if it were to work for 13 hours daily?

Ans. 281·5 quarters.

Ex. 114.—Water is raised to a height of 54 ft. above the bottom of a fall of water of 14 ft. down which descend 200 cubic feet per minute, by means of a wheel worked by the fall, and having a modulus 0·60; how many cubic feet of water will be thus raised per minute, and if the water were raised from the top of the fall to the same point, what would the number of cubic feet then be? *Ans.* (1) 31·1 cub. ft. (2) 34·7 cub. ft.

[Of course in the second case the number of cubic feet of water taken from the top of the fall being x, the number of feet that turn the wheel will be $200-x$.]

Ex. 115.—Water has to be raised from a mine 120 ft. deep, the whole of the water raised forms a stream with a fall of 30 ft. the machinery by which the water is raised is a steam engine of 50 horse-power, and an overshot wheel whose modulus is 0·715 turned by the stream; determine the whole number of cubic feet raised per minute. *Ans.* 267·8 cub. ft.

Ex. 116.—In the last example if the ground allowed an exit to be made for the water 30 feet below the mouth of the shaft (by which of course the fall is entirely lost), what must be the horse-power of the engine to raise per minute the same amount of water as before? *Ans.* 45·6 H.-P.

16. *The work of living agents.*—The efficiency of men and animals is estimated in the same manner as that of the inanimate agents already considered, viz. by the number of units of work they are capable of yielding. The number yielded under given circumstances by any particular agent must of course be determined by experiment. The results of experiment on this matter are registered in the tables that follow; they are based on similar tables given in General Morin's *Aide-Mémoire.* It must be borne in mind that these tables give mean results when the agent works in the best manner. It would be very possible for the agents to work with greater velocities than those assigned, but were this done they would yield a much smaller daily amount of work — compare the work done by a horse walking with that done by a horse trotting.

TABLE VIII.

WORK DONE BY MEN AND ANIMALS.

Nature of Labour.	Daily duration of work in hours.	No. of units of work per day.	No. of units of work per min.	Weight raised or mean pressure	Velocity. Ft. per min.	Velocity. Miles per hour
(1) *Raising weights vertically.*						
A man mounting a gentle incline or ladder, without burden, *i. e.* raising his own weight	8·0	2032000	4230	145	29	0·33
Labourer raising weights with rope and pulley, the rope returning without load	6·0	563000	1560	40	39	0·44

TABLE VIII. (*continued.*)

Nature of Labour.	Daily duration of work in hours.	No. of units of work per day.	No. of units of work per min.	Weight raised or mean pressure	Velocity.	
					Ft. per min.	Miles per hour
Labourer lifting weights by hand	6·0	531000	1480	44	34	0·38
Labourer carrying weights on his back up a gentle incline or up a ladder and returning unladen	6·0	406000	1130	145	8	0·09
Labourer wheeling materials in a barrow up an incline of 1 in 12 and returning with the empty barrow	10·0	313000	520	130	4	0·045
Labourer lifting earth with a spade to a mean height of $5\frac{1}{4}$ feet	10·0	281000	470	6	78	0·9
(2) *Action on Machines.*						
Labourer walking and pushing or pulling horizontally	8·0	1500000	3130	27	116	1·32
Labourer turning a winch	8·0	1250000	2600	18	144	1·64
Labourer pulling and pushing alternately in a vertical direction	8·0	1146000	2390	11	216	2·70
Horse yoked to a cart and walking	10·0	15688000	26150	150	175	2·00
Do. to a whim gin	8·0	8440000	17600	100	175	2·00
Do. do. trotting	4·5	7036000	26060	$66\frac{2}{3}$	391	4·44
Ox yoked to a whim gin and walking	8·0	8127000	16930	145	117	1·33
Mule do. do.	8·0	5627000	11720	$66\frac{2}{3}$	176	2·00
Ass do. do.	8·0	2417000	5030	30	168	1·95

The following table gives the useful effect of men and animals employed in the horizontal transport of burdens. The second and third columns give the useful effect, viz. the product of the weight in lbs. and the distance in feet. The reader must not mistake this for the units of work done by the agent, the agent being employed *not* in raising the weight, but in overcoming the passive resistances, friction, &c. which depend on the weight indeed, but are only a fraction of it.

TABLE IX.

USEFUL EFFECT OF AGENTS EMPLOYED IN THE HORIZONTAL TRANSPORT OF BURDENS.

Agent.	Duration of daily work.	Useful effect daily.	Useful effect per minute.	Weight transported.*	Velocity.	
					Ft. per min.	Miles perhour.
Man walking on a horizontal road without burden *i. e.* transporting his own weight	10·0	25398000	42330	145	292	3·32
Labourer transporting materials in a truck on two wheels, returning with it empty for a new load	10·0	13025000	21710	220	99	1·12
Do. do. in a wheelbarrow	10·0	7815000	13030	130	160	1·14
Labourer walking with a weight on his back	7·0	5470000	13030	90	145	1·64
Labourer transporting materials on his back and returning unburdened for a new load	6·0	5087000	14110	145	97	1·10

* Exclusive of the weight of the barrow, truck, cart, &c. Poncelet, Mec. Ind. p. 247.

TABLE IX. (*continued.*)

Agent.	Duration of daily work.	Useful effect daily.	Useful effect per minute.	Weight transported.*	Velocity.	
					Feet per min.	Miles per hour.
Do. do. on a hand-barrow	10·0	4298000	7160	110	65	0·74
Horse transporting materials in a cart, walking, always laden	10·0	200582000	334300	1500	223	2·53
Do. do. trotting	4·5	90262000	334300	750	44	5·06
Do. transporting materials in a cart returning with the cart empty for a new load	10·0	10940800	182350	1500	121	1·38
Horse walking with a weight on his back	10·0	34385000	57310	270	212	2·41
Do. do. trotting	7·0	32092000	76410	180	424	4·82

Ex. 117.—How many men would be required to raise by means of a capstan an anchor weighing 1 ton from a depth of 30 fathoms, in 15 minutes? *Ans.* 9 nearly.

Ex. 118.—In what time would 20 men raise the anchor in the last example? *Ans.* 6·4 min.

Ex. 119.—Through how great a distance would 30 men raise the anchor in Ex. 117 in each minute. *Ans.* 42ft. nearly.

Ex. 120. There is a well 150 ft. deep, a labourer raises water from it by a rope and pulley, how many cubic feet of water will he raise in a day? *Ans.* 60 cub. ft.

Ex. 121.—How many cubic feet of water would a steam engine of 10 horse-power raise from this well in 24 hours? How many labourers would be required to do the same amount of work if they raised the water by wheel-and-axles, and how many if they raised it by means of capstans? How many horses would do the same amount of work walking in whim gins?

Ans. (1) 50688 cubic feet. (2) 380 labourers. (3) 317 labourers. (4) 56 horses.

Ex. 122.—In how many minutes could 20 men working on a capstan raise an anchor weighing 2 tons from a depth of 200 fathoms? *Ans.* 85·88 min.

Ex. 123.—How many men would in 40 minutes raise the anchor in the last example? *Ans.* 43 men.

Ex. 124.—Through how many fathoms could 15 men raise the anchor of Ex. 122 in 10 minutes? *Ans.* $17\frac{1}{2}$ nearly.

Ex. 125.—If 13 men are required to raise an anchor through 180 fathoms in 20 minutes, what must be the weight of that anchor? *Ans.* $753\frac{1}{3}$ lbs.

Ex. 126.—A town is situated 25 miles from the mouth of a coal pit, from which coal is taken to the town by a level railway on which the resistance is 10 lbs. per ton; the engine employed is of 15 horse-power and weighs with its tender 10 tons; each truck weighs 3 tons and contains 7 tons of coals; on each journey the engine takes 5 full trucks and returns with 5 empty trucks; supposing no time to be lost at the ends of the journey, how many tons of coals will be taken to the town in 48 hours? How many horses would be required to convey the same quantity of coals? *Ans.* (1) 455 tons. (2) 679 horses.

17. *Remarks on the work yielded by different agents.*—The following remarks upon the preceding tables and examples are worthy of the attention of the reader.

(1.) Every agent must be allowed to move at a certain rate in order to do the greatest amount of work it is capable of yielding; thus, a horse walking does considerably more work than a horse trotting, as an inspection of the tables will show. And this is true not of animate agents only, but of inanimate; thus the work yielded by the consumption of a given quantity of coal will be larger in the case of a slow than of a fast engine.

(2.) Also, in order that an animate agent may do its greatest amount of work it must not be required to exert more than a certain amount of pressure. This is also plain from an inspection of the table.

(3.) It follows from the above considerations that though two agents may be capable of doing the same work in the same time, it may be in practice impossible or disadvantageous to substitute the one for the other. Thus an ox and a horse walking in a whim gin do very nearly the same amount of work; but since the ox moves more slowly, and exerts a greater pressure than the horse, it would generally be disadvantageous to substitute a horse for an ox in a

machine requiring a slow heavy pressure. Again, in cases where great speed is a *desideratum*, it would generally be impossible by any machinery to make the slow agent perform the labour of the rapid agent; as, for instance, in the case of locomotion.

18. *On the Cost of Labour.*—The chief elements in the cost of labour may be enumerated as follows: —

(1.) In the case of human labour, the whole cost is the wages paid.

(2.) In the case of a horse, the elements of expense are attendance, keep, and the original cost; the last is but a small portion of the expense. Thus, if we suppose a horse to cost 20*l.* and to continue in working order for ten years, and reckon the value of money at four per cent. per annum, the element of cost would be 2·465*l.* yearly, or not quite 1*s.* per week.

(3.) In the case of a steam engine, the chief elements are the original cost and subsequent repairs, attendance, and fuel. Of these elements the most important is that of fuel; and accordingly there is a special definition of the power of an engine with reference to the consumption of fuel. The definition is as follows: —

Def.—The number of units of work yielded by an engine in consequence of the consumption of 1 bushel (*i. e.* 84 lbs.) of coal, is called the *duty* of that engine.

The extent to which the economy of fuel may be carried is very remarkably illustrated by the engines employed to drain the mines in Cornwall. In 1815, the average duty of these engines was 20 millions; in 1843, by reason of successive improvements, the average duty had become 60 millions, effecting a saving of 85,000*l.* per annum*; it is

* Bourne on the Steam Engine p. 171. It may be remarked that this result depends largely on the construction of the boiler; 1 lb. of coal in the Cornish boiler evaporates $11\frac{1}{2}$ lbs. of water, while in the waggon shaped boiler 8·7 is the maximum. Fairbairn, Useful Information p. 177.

Find the pressure per square inch on the piston, the horse-power (as measured by pressure of steam) and the duty of the engine.

Ans. (1) 12·75 lbs. (2) (nearly) 42 H.-P. (3) 133,000,000 duty.

Ex. 133.—In Ex. 126 suppose the engine and trucks on the one hand and the horses and carts on the other to want renewal every ten years; suppose also that each horse and cart costs 40*l.*, that one man attends to every six horses and is paid 3*s.* a day, that each horse's keep is 1*s.* 6*d.* a day, that there are two turnpikes on the road at each of which there is a toll of 6*d.*; determine the cost of transporting 455 tons of coals. Next suppose the engine and tender to cost 1000*l.*, each truck 120*l.* (15 trucks are required to prevent loss of time); that there are three drivers and three stokers each at 6*s.* a day; that money is worth 5 per cent. and that each mile of road cost 10,000*l.* to make and 365*l.* a year to keep in repair; determine in this case the cost of transporting 455 tons of coals. Also if coal cost 3*s.* a ton at the pit mouth what will it cost in the town according to each method of transport neglecting profit.

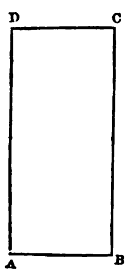
Ans. (1) 218*l.* (2) 123*l.* (3) 12*s.* 6*d.* a ton by cart.
(4) 8*s.* 5*d.* a ton by rail.

[Interest on the cost price of engine, trucks, horses and carts can be neglected.]

SECTION II.

19. *On the work done by a variable pressure.*—There are two important questions in the subject of work which we shall treat in the present section: they are (1) the work done by a variable pressure, when exerted through a certain space; (2) the total amount of work done in raising a number of weights through different heights.

Fig. 5.



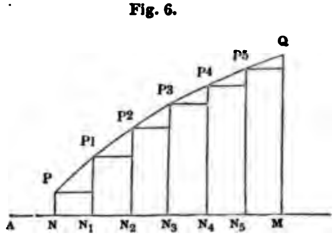
As an introduction to the theorem which follows, it may be remarked, that if a constant pressure of P lbs. act through a space of S feet, and if a rectangle A B C D be drawn, of which the base A B represents the S feet on scale, and the perpendicular A D represents the P lbs. on the same scale: then, since the area of A B C D contains P S square units on the same scale, that area will correctly represent the work done by P.

Proposition 1.

If a variable pressure acts through a certain space, and if a curve be drawn the abscissæ of which represent the spaces, and the corresponding ordinates the pressures at the ends of those spaces, then will the area of the curve between any two ordinates represent the work done by the pressure while acting through a space represented by the difference between the extreme abscissæ.

When the pressure has acted through a space represented on a certain scale by A N, suppose it to be represented on the same scale by PN; also, when it has acted through a space A M, suppose it to be represented on the scale by Q M; let the curve P Q be drawn in such a manner that any ordinate P_3N_3 represents the pressure when it has acted through a space A N_3; we have to prove that the area P N M Q represents the work done by the pressure while acting through the space N M.

Fig. 6.



For divide N M into any number of equal parts in N_1, N_2, N_3 .... draw the ordinates P_1N_1, P_2N_2, P_3N_3.... and complete the rectangles PN_1, P_1N_2, P_2N_3.... Now, if we suppose the pressures at the beginning of each of these short spaces N N_1, N_1N_2, N_2N_3.... to continue unchanged during its action through that short space, we shall nearly represent the actual case, and shall represent it more nearly the smaller we take the spaces, the actual case being the limit we continually approach by increasing the number of spaces.

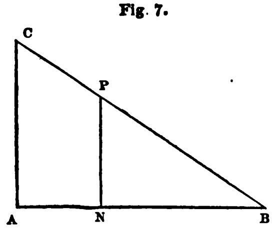
But if the pressure acts uniformly through each space, it will do a number of units of work represented by the sum of the rectangular areas PN_1, P_1N_2, P_2N_3 ....., and this being true whatever be the number of the small spaces,

the work actually done will be properly represented by the limit of the sum of these rectangles, *i.e.* by the curvilinear area PN MQ.

Cor.—It must be borne in mind that the scale must be the same for lbs. and for feet; thus, if the scale be in inches, then P N must be as many inches long as the pressure contains lbs., and N M must be as many inches long as the space represented contains feet; this being so, the area of the curve in square inches will give the number of units of work.

Ex. 134.—A rope l ft. long and weighing w lbs. per foot hangs by one extremity, determine the number of units of work required to wind up a ft. of the length.

Fig. 7.



Take A B on scale equal to l, draw A C perpendicular to A B and on the same scale equal to $w\,l$, join BC; in AB take any point N, draw PN parallel to AC, then since

$$PN : NB :: CA : AB :: w : 1.$$

Therefore $PN = w\,NB$, *i.e.* the ordinate P N represents on scale the weight of the rope left hanging when the extremity has been raised through a space A N. Hence the area A B C represents the number of units of work required to wind up the whole rope, and the area C B P N the number of units of work required to wind up a length A N of the rope. Hence if U is the required number of units

$$U = wa\,(l - \frac{a}{2}).$$

Ex. 135.—A weight of 2 cwts. has to be raised from a depth of 100 fathoms by a rope 3 in. in circumference; determine the number of units of work that must be expended in raising it and the number of minutes in which 8 men would do the work, by means of a capstan.

Ans. (1) 207300 units. (2) 8·28 min.

Ex. 136.—How heavy will that anchor be which 13 men will raise by means of a capstan from a depth of 180 fathoms in 40 min., supposing the cable to weigh 1125 lbs.? (neglecting the buoyancy of the water).

Ans. $945\frac{1}{2}$ lbs.

Ex. 137.—A chain each foot of which weighs 8 lbs. is suspended from the top of a shaft the depth of which is 50 fathoms; determine the number of units of work required to wind up each successive 100 ft. of its length;

determine also the length of the chain which will require twice as many units of work to wind it up.

Ans. (1) 200,000, 120,000, 40,000 units of work respectively. (2) 424 ft.

Ex. 138.—If a chain 300 ft. long and weighing 8 lbs. per foot is wound up in 4 min.; how many men working on a capstan, would do it? How many horses working in a whim gin? How many steam horses? How many of each agent would be required if the weight per foot of the chain were doubled? And how many if the length of the chain were doubled?

Ans.	(1) 29 men.	(2) 5·1 horses.	(3) $2\frac{8}{11}$ horse-power.
	(4) 57 men.	10·2 horses.	$5\frac{5}{11}$ horse-power.
	(5) 115 men.	20·4 horses.	$10\frac{10}{11}$ horse-power.

Ex. 139.—A chain is a ft. long, divide it into n parts such that the winding up of each may require the same number of units of work.

Ans.

$$\frac{a}{\sqrt{n}}(\sqrt{n}-\sqrt{n-1}),\ \frac{a}{\sqrt{n}}(\sqrt{n-1}-\sqrt{n-2}),\ \frac{a}{\sqrt{n}}(\sqrt{n-2}-\sqrt{n-3}).$$

Ex. 140.—Coal is raised from the bottom to the mouth of a pit 150 ft. deep in loads of a quarter of a ton, the box containing it weighs 1 cwt., the rope by which it is raised is 3 in. in circumference; determine the number of units of work spent in raising the coal, and the number spent in raising the box and rope. Suppose the lifting engine to work with 10 horse-power determine the weight of coals raised in 2 hours, supposing the ascent and descent of the box take to equal times.

Ans. (1) 84000 units to raise coal. (2) $21356\frac{1}{4}$ units to raise box and rope. (3) 47 tons.

Ex. 141.—In the last example suppose machinery to be employed by means of which the same drum winds up the rope of an ascending box and unwinds that of a descending box. Determine the number of tons raised in 2 hours? * *Ans.* 118 tons.

[Of course the units of work done by the descending box and rope will nearly equal that expended on the ascending box and rope—the weight of box and rope can therefore be neglected.]

Ex. 142.—Determine the number of tons raised under the conditions of Ex. 140 and 141 supposing $\frac{1}{2}$ a minute is expended in filling or emptying the box. *Ans.* (1) 18 tons. (2) $39\frac{3}{4}$ tons.

Ex. 143.—If 4 cwt. of material are drawn from a depth of 80 fathoms by a rope 5 in. in circumference; how many units of work are expended in raising it, and what horse power is necessary to raise it in $4\frac{1}{2}$ minutes?

Ans. (1) 344,640 units. (2) 2·32 H.-P.

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* The primary object of this mode of working was, probably, to save time, the saving of labour being an accidental result; though that saving is very considerable.

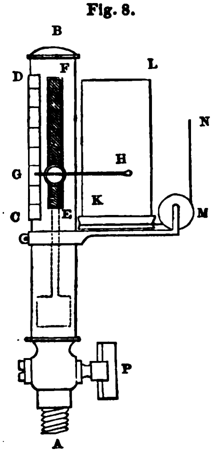
Ex. 144.—A rope 3 in. in circumference is strong enough to bear a working strain of 4 cwt.; how many units of work are wasted in the last example by using a rope 5 in. in circumference. *Ans.* 82944 units.

Ex. 145.—A winding engine raises to the surface a load of 12 cwts. in $6\frac{1}{2}$ minutes from a depth of 115 fathoms; the rope employed is a flat rope composed of 3 ropes each 3 in. in circumference. What is the horse-power of the engine? *Ans.* 5·67 H.-P.

Ex. 146.—If the engine in the last example has a cylinder 20 in. in diameter, and makes per minute 15 strokes of 2 ft. 10 in., under what mean pressure per square inch of steam does it work if its modulus is 0·55? *Ans.* 25·5 lbs.

20. *The Steam Indicator.*—A very instructive application of Proposition 1 occurs in the steam indicator, which may be sufficiently described as follows: A B is a small hollow cylinder containing a powerful spring, which can be partly seen through the aperture E F, within the indicator is a small piston or plunger (marked in the figure by dotted lines) which is kept down by the spring, so that if it is forced up the compression of the spring gives the amount of the compressing force, which can be read off on the scale C D by means of the pointer G H, which rises and falls with the plunger. The end H of the pointer carries a pencil, the point of which rests against a sheet of paper wrapped round a cylinder K L; if this cylinder is stationary, and the pencil moves, a vertical straight line will be described; if the pencil is stationary, and the cylinder revolves, a horizontal straight line will be described; but if both the pencil moves and the cylinder revolves, a curved line will be described.

Fig. 8.

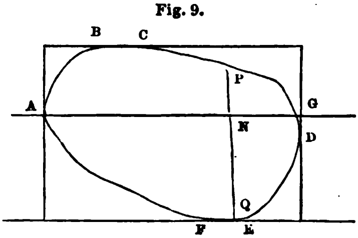


The instrument is used in the following manner:—The end A being screwed into an aperture properly constructed, the steam in the interior of the cylinder of the

steam engine can be admitted into the indicator by opening the cock P; at first, however, the cock P is shut, so that the pointer remains stationary. The end of the string M N is attached to some part of the engine in such a manner that the paper K L makes one revolution while the piston of the steam engine makes a stroke; this being done, and the cock kept shut, the pencil will trace on the paper a straight line, called the atmospheric line: on the next stroke the cock is opened, and now the steam pressing on the plunger the pencil will rise or fall according as the pressure of the steam is greater or less than that of the atmosphere, and will describe a curve that will return into itself at the end of a double stroke (or revolution). The area of the curve thus described will give the amount of work done by the steam during a *single* stroke.

To explain this, suppose A B C D E F to be the curve given by the indicator, A G the atmospheric line, draw P N Q any double ordinate, then P N represents the excess of the steam pressure above that of the atmosphere when the ascending piston is at a certain point, and N Q represents the defect of the vacuum pressure below that of the atmosphere when the descending piston is at the same point.

Fig. 9.



Now the effective pressure of the steam is the excess of the steam pressure above the vacuum pressure; but

$$P\,N = \text{steam pressure} - \text{atmospheric pressure},$$
$$N\,Q = \text{atmospheric pressure} - \text{vacuum pressure},$$
$$\therefore P\,N + N\,Q = \text{steam pressure} - \text{vacuum pressure},$$

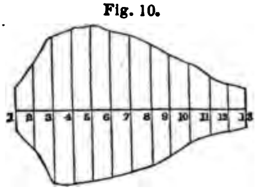
therefore P Q represents the effective pressure of the steam when the ascending piston is at the point corresponding to N, *i.e.* assuming the vacuum pressure at any point of

one stroke to be the same at the same point of the next stroke. If, then, for the sake of distinctness*, we suppose each inch of the ordinate to denote a pressure of 1 lb. and each inch of the abscissa (*i. e.* of the atmospheric line) to denote a foot of the stroke, the area of the curve will give the number of units of work done during a *single* stroke by the steam on an area equal to that of the plunger, and if the area of the piston of the steam engine be n times that of the plunger, the work done by the steam during a *single* stroke will be n times that given by the curve.

The area of the curve may be found by Simpson's rule, viz.—Divide A G into any number of equal parts, and draw the corresponding ordinates; take the sum of the extreme ordinates, four times the sum of the even ordinates, and twice the sum of the odd ordinates, add them together, and multiply the sum by one of the parts of the abscissa; the product will be three times the area of the curve.†

Ex. 147.—Let the curve shown in the figure to be that given by a stroke of 3 feet; suppose A B to be divided into 12 equal parts, and the ordinates 1 . 2 . 3 ..... to be drawn; suppose that they represent respectively pressures per square inch of 2, 4·73, 8·05, 8·15, 8·11, 8·09, 7·93, 7·02, 5·75, 4·61, 3·84, 3·02, 2·51 lbs. respectively. The radius of the piston being 15 in. determine the number of units of work done per stroke; and the *mean* pressures per square inch on the piston.

Fig. 10.



[The *mean* pressure per square inch must, of course, do the same number of units of work per stroke as the actual pressures.]

Ans. (1) 12627 units. (2) 5·95 lbs. per square inch.

Ex. 148.—Determine the number of units of work and the mean pressure per square inch on a piston $3\frac{1}{2}$ feet in diameter having a stroke of 5 feet, if the ordinates measured at intervals corresponding to three inches of the stroke give the following pressures 5·03, 12·57, 18·04, 20·73, 21·03,

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* In practice the scale would be considerably less than this.

† The curve given by the indicator is useful in other ways beside that mentioned in the text. Bourne on Steam Engine, p. 246.

21·11, 21·25, 20·72, 20·14, 18·63, 15·45, 13·24, 10·83, 8·53, 6·49, 4·87, 3·99, 3·74, 3·52, 3·25, 2·75.

Ans. (1) 87604 units. (2) 12·65 lbs. per sq. in.

21. *Work expended on the elongation of bars.* — It is plain that if a rod be lengthened by a gradually increasing pressure, the pressure at any degree of elongation will be proportional to that elongation; so that if the abscissæ represent the degree of elongation, and the ordinates the strain, the area which gives the units of work will be a triangle. Hence:

Ex. 149.—There is a bar the length of which is L and section K; it is gradually elongated by a length l; if its modulus of elasticity be E, show that the work expended on its elongation will be given by the formula

$$U = \frac{l^2}{2L} KE.$$

Ex. 150.—The pumping apparatus of a mine is connected with the engine by means of a series of wrought iron rods 200 ft. long; the section of each rod is $\frac{3}{4}$ of a square inch; the strain is estimated at 6 tons; how many units of work are expended at every stroke upon the elongation of the bars?

Ans. 830 units.

Ex. 151.—A bar of wrought iron 100 ft. long with a section of 2 square inches has its temperature raised from 32° F. to 202° F.; how many units of work has the heat done? *Ans.* 3875 units.

22. *The work expended in raising weights through various heights.*— The questions arising out of this important part of the present subject are solved by means of the following proposition.

Proposition 2.

When any weights are raised through different heights, the aggregate of the work expended is equal to the work that would be expended in lifting a weight equal to their sum through the same distance as that through which the centre of gravity of the system of weights has been raised.

Let W_1, W_2, W_3 ..... be the weights of each separate

body; conceive a horizontal plane to pass below them all; let $h_1, h_2, h_3 \ldots\ldots$ be the heights of these bodies above the plane before they are lifted, and let H be the height of the common centre of gravity; then (Prop. 15)

$$H(W_1+W_2+W_3\ldots\ldots) = W_1h_1+W_2h_2+W_3h_3+\ldots \quad (1)$$

Also, let $k_1, k_2, k_3 \ldots\ldots$ be the heights of these weights respectively, after they have been lifted, and K the height of their common centre of gravity; then

$$K(W_1+W_2+W_3 \cdots\cdots) = W_1k_1+W_2k_2+W_3k_3+\ldots \quad (2)$$

hence, subtracting (1) from (2), we obtain

$$(K-H)(W_1+W_2+W_3+\ldots) = W_1(k_1-h_1)+W_2(k_2-h_2)+W_3(k_3-h_3)\ldots \quad (3)$$

Now, $W_1, W_2, W_3 \ldots$ are severally raised through the heights k_1-h_2, k_2-h_2, $k_3-h_3 \ldots\ldots$; therefore the right hand side of equation (3) gives the aggregate work expended on lifting them; hence that work is equal to

$$(K-H)(W_1+W_2+W_3\ldots),$$

i.e. to the work that must be expended on lifting a weight $W_1+W_2+W_3+\ldots$ through a height $K-H$.—Q. E. D.

Cor.—In the case of the transport of bodies along any parallel line, the principle enunciated in the theorem will hold good, since the resistances are in a constant ratio to the weights.

Ex. 152.—How many units of work must be expended in raising the materials for building a column of brickwork 100 ft. high and 14 ft. square; and in how many hours will an engine of 2 horse-power raise them? *Ans.* (1) 109,760,000 units. (2) 27·71 hours.

[Since the material has to be raised from the *ground*, the common centre of gravity will have to be raised from the ground to the centre of gravity of the column, *i.e.* to its middle point 50 feet above the ground.]

Ex. 153.—A shaft has to be sunk to a depth of 130 fathoms through chalk; the diameter of the shaft is 10 ft.; how many units of work must be expended on raising the materials? In how long a time could this be

done by a horse walking in a whim gin? How many men working in a capstan would do it in the same time? Determine the expense of the work supposing the horse to cost 3*s*. 6*d*. a day, and the wages of a labourer to be 2*s*. 6*d*. a day.

Ans. (1) 3457 million units. (2) 409·6 days. (3) 5·62 men.
(4) Cost of horse 71*l*. 14*s*. Cost of men 287*l*. 15*s*.

Ex. 154.—If the work in the last Example is to be done in 24 weeks by a steam engine working 8 hours a day, 6 days a week, what must be the horse-power of the engine? *Ans*. 1·521 H.-P.

Ex. 155.—In Ex. 153 suppose the box in which the material is raised to weigh $\frac{1}{2}$ cwt., the rope to be 3 in. in diameter, and each load to be 4 cwts. of chalk, also suppose the box to take as long in ascending as in descending and that $\frac{1}{4}$ of a minute is lost in unhooking and hooking at the bottom of the shaft and the same at the top; when the shaft is 100 ft. deep determine the time that elapses between the starting of one load and the starting of the next; the engine working at $1\frac{1}{2}$ horse-power. *Ans*. 2·62 min.

Ex. 156.—Determine the same as in the last Example when the shaft is x ft. deep.

$$Ans.\ \frac{112x + 0{\cdot}045x^2}{5500} + 0{\cdot}5 \text{ min.}$$

Ex. 157.—Determine the whole time of raising the materials of the shaft in Ex. 153 under the conditions of Ex. 155. *Ans*. 3331 hours.

Ex. 158.—Referring to Ex. 153, 5, suppose the drum of the winding machine to have two ropes wound round it in contrary directions so that it unwinds one rope while winding up the other, and that consequently an empty box descends while a full one is being raised (as in Ex. 141); determine the time that must elapse between two consecutive lifts of 4 cwts. when the shaft is 100 ft. deep. *Ans*. 1·155 min.

Ex. 159.—Obtain a determination similar to that in the last example, when the shaft is x ft. deep.

$$Ans.\ \frac{448x}{49500} + 0{\cdot}25 \text{ min.}$$

Ex. 160.—Obtain the whole time of lifting the materials from the shaft under the circumstances of Ex. 158. *Ans*. 1246 h. 24 min.

Ex. 161.—In how long a time would a 15 horse-power engine empty a shaft full of water the diameter of the shaft being 8 ft. and the depth 200 fathoms? If the engine has a duty of 30 millions determine the amount of coal consumed in emptying the shaft.

Ans. (1) 76 hours. (2) 75·4 bushels.

Ex. 162.—There is a certain railway 200 miles long; it may be assumed that in the course of 10 years there will be 50,000 tons of iron railing laid down; and that it will be equally distributed along the line. How many units of work must be expended in conveying the rails (neglecting the weight of the trucks), if the depot is at one end of the line? And how

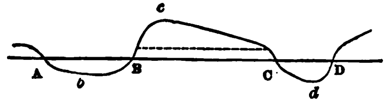
many if the depot is in the middle of the line? The resistances being reckoned at 8 lbs. per ton.

Ans. (1) 211,200 million units. (2) 105,600 million units.

Ex. 163.—How many journeys of 200 miles performed by a train weighing 50 tons does the difference of the results in the last Example represent? Resistances 8 lbs. per ton. *Ans.* 250 journeys.

23. In the construction of a railway it is endeavoured that the volume of the embankments shall equal the volume of the cuttings and tunnels, so as to avoid, on the one hand, the necessity of excavating for the mere purpose of obtaining earth, and, on the other, of heaping waste earth. Besides this, it is further endeavoured (by a proper arrangement of gradients) that the volume of a given embankment shall equal the volume of its neighbouring cutting, so that there may be no need of transporting earth from a long distance. Let us suppose a railway to be so arranged that the volume of each embankment equals the volume of its next cutting; now, this railway may be constructed on either of two principles, viz. (1) the labourers might commence at one end of a cutting, and work through to the other end of it, forming with the earth so obtained the whole of one of the neighbouring embankments; (2) there may be two parties, one beginning at one end of the cutting and the other at the other end, and forming a part of each of the neighbouring embankments. The latter mode evidently saves time; it also saves labour. The following example illustrates the economy of labour thus

Fig. 11.



effected. In practice, the method cannot be adopted strictly, and the actual gain will not be so great as would appear from the example.

Ex. 164.—Let *a b c d* represent the section of the country through which passes a railway whose level is A B C D; the hill is perforated by a tunnel B C, 4 miles long, the area of the section of which is 350 square feet; the content of this tunnel is equal to that of either of the embankments A *b* B or C*d*D; A B is 3 miles long, C D is 2 miles long, and have their centres of gravity in their middle point. The centre of gravity of half the embankment B *b* is distant $\frac{7}{8}$ of a mile from B, and that of half the embankment C *d* D is distant $\frac{5}{8}$ of a mile from C. The specific gravity of the earth is 2·4; the resistances 10 lbs. per ton; the weight of the trucks is neglected. Determine the number of units of work expended on the transport of material (1) when the labourers begin at B and work to C forming the embankment B *b* A; (2) when they begin at C and work to B forming the embankment C *d* D; (3) when they begin simultaneously at B and C and form half of each of the embankments.

Ans. (1) 91,476 million units.
(2) 78,408 million units.
(3) 45,738 million units.

CHAP. III.

THE FUNDAMENTAL PRINCIPLES OF STATICS.

24. *Mechanics.*—The science of Mechanics is that which treats of *the motion and rest of bodies as produced by force.* The words "as produced by force" are added in order to exclude the science of *pure motion* or *mechanism*, which treats of the *forms* of machines, and in which machines are regarded merely as modifiers of *motion.* Into all questions which are properly mechanical the idea of *force* must enter.

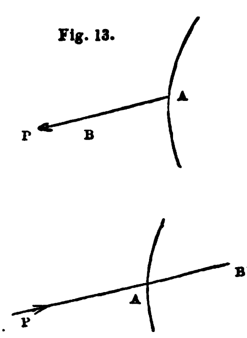
25. *Force.*—Force may be defined to be any cause which puts a body in motion, or which tends to put a body in motion when its effect is hindered by some other cause. On this definition the following remark is to be made:—Suppose a given weight (say of 12 lbs.) is supported by a string passing over a pulley and fastened at one end; next, suppose an equal weight to be supported by a man's hand; lastly, suppose an equal weight to be supported by the expansive pressure of a spring. Now, here we have three physical agents, viz. the tension of a string, the muscular power of a man, and the elastic power of a spring, very different in many respects, but agreeing in their common capacity to support a given weight. They may clearly be regarded as equal, when viewed with reference to that capacity; and in the case we have supposed, each may be correctly represented by 12 lbs. In short, as in geometry, we regard all bodies as equal which can successively fill the same space, without any regard to their physical qualities, such as weight, colour, &c. so in mechanics we regard all forces as equal which will severally

balance by direct opposition the same weight, irrespectively of their *physical origin.*

26. *Statics and Dynamics.*—It follows, from the definition, that, in Mechanics, we can consider a force either as producing motion, or as concurring with others in producing rest. Accordingly, the science of mechanics is divided into two distinct though closely connected branches, viz. statics and dynamics. Of these, statics is that science which determines the conditions of the equilibrium of any body or system of bodies under the action of given forces. Dynamics is that science which determines the motion, or the change of motion, that ensues in a body or system of bodies subjected to the action of a force or forces that are not in equilibrium.

27. *Determination of a pressure.*— From what has already been said, it appears that the *magnitude* of any pressure is assigned by considering the weight it would just support if applied directly upward; in other words, we arrive at the magnitude of any pressure by comparing it with the most familiar and measurable of pressures, viz. weight. A little consideration will show us that the effect of a pressure in any case depends not only on its amount but also on its point of *application*, and the *line* along which it acts. We may say, therefore, in general terms, that a pressure is completely determined when we know (1) its magnitude, (2) its point of application, (3) the direction of its action. A *line* is frequently said to *represent* a pressure; when this is the case, it must be drawn *from* the point of application of the pressure along the line of its action, and contain as many units of length (say inches) as

Fig. 13.



the pressure contains units of weight (say lbs.). It is of great importance that the student should attend to all the conditions which must meet when a line correctly represents a pressure. Suppose a pressure of P lbs. to act on a body at the point A; if the pressure is a *pull*, as in the first figure, the line A B containing as many inches as P contains lbs. will represent the pressure; but if the pressure is a *push*, A B must be measured, as on the second figure.

28. *Resultant and components.*—If we consider any system of pressures that keep a body in equilibrium, it is plain that any one of them balances all the others: thus, if three strings be knotted together, and be pulled by pressures of P lbs., Q lbs., and R lbs. respectively, it is plainly a matter of indifference whether we consider that P balances Q and R, or that Q balances R and P, or that R balances P and Q. Let us consider that R balances P and Q; now R would of course balance a pressure R′ exactly equal and opposite to itself; so that if we substitute R′ for P and Q, or *vice versa*, P and Q for R′, in either case R is balanced, and the force R′ is equivalent to P and Q; under these circumstances, R′ is called the resultant of P and Q, and P and Q are called the components of R′. Hence we may state generally,

Def.—That pressure which is equivalent to any system of pressures, is called their resultant.

Def.—Those pressures which form a system equivalent to a single pressure, are called its components.

29. *Resultant of pressures acting along the same straight line.*—If the pressures act in the same direction the resultant must be their sum. If some act towards the right and some towards the left, the first set can be formed into a single pressure (P) acting towards the right, the second set can be formed into a single pressure (Q) acting

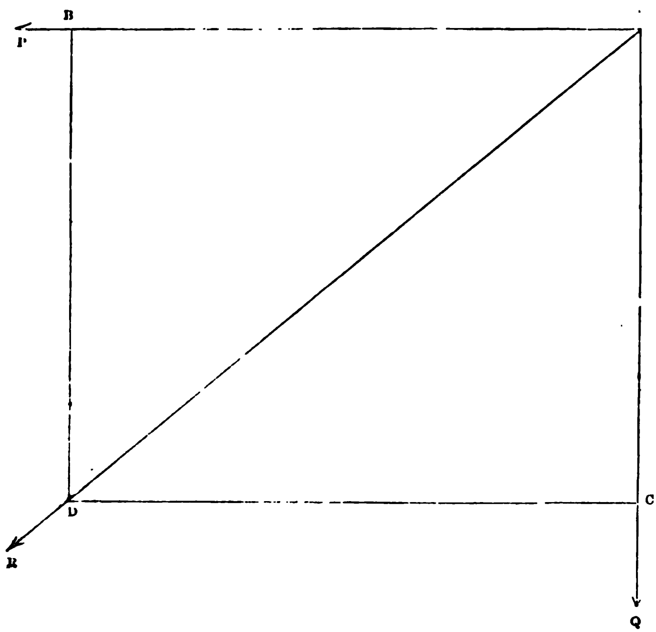


Fig. *a.* p. 51.

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towards the left: the resultant of these two, and therefore of the original set of pressures, will be equal to the difference between P and Q, and will act in the direction of the greater.

Ex. 165.—If three men pull on a rope to the right with pressures of 31 20, and 27 lbs. respectively, and are balanced by 2 men who pull with pressures of 40 and P lbs. respectively; find P. *Ans.* 38 lbs.

Ex. 166.—In the last example find the resultant of the 5 pressures (1) if P = 30 lbs.; (2) if P = 40 lbs.

Ans. (1) 8 lbs. acting towards the right.
(2) 2 lbs. acting towards the left.

Ex. 167.—There is a rope AB and men pull along it in the following manner: the first with a pressure of 50 lbs. towards A; the second with a pressure of 37 lbs. towards B; the third with a pressure of 35 lbs. towards A; the fourth with a pressure of 20 lbs. towards A; the fifth with a pressure of 54 lbs. towards B; the sixth with a pressure of 27 lbs. towards A; the seventh with a pressure of 52 lbs. towards B; the eighth with a pressure of 30 lbs. towards B. Determine the single pressure that must act along AB to balance them and find whether it acts towards A or B.

Ans. 41 lbs. acting towards A.

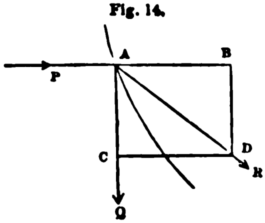
30. *The parallelogram of pressures.*—When two pressures act on a point along different lines, their resultant is determined by the following rule, which is denominated the principle of the parallelogram of pressures:—*If two pressures act on a point, and if lines be drawn representing those pressures, and on them as sides a parallelogram be constructed, that diagonal which passes through the point will represent the resultant of the pressures.* The student, when applying this principle to any particular case, must bear in mind the meaning of the words a *line represents a pressure* (Art. 27).

Ex. 168.—If at a point A of a body two ropes AP and AQ are fastened and are pulled in directions AP, AQ at right angles to each other by pressures of 120 and 100 lbs. respectively; determine the magnitude and direction of the resultant pull on the point A. (See fig. *a.*)

Along AP measure on scale AB containing 120 units of length, and along AQ measure AC containing 100 units of length; complete the rectangle BC and draw the diagonal AD; this line represents the magnitude

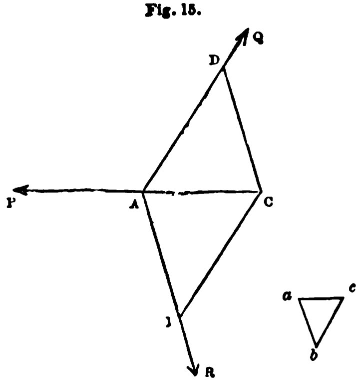
and direction of the resultant. In fig. (a) the scale employed is 1 in. for 40 lbs.; the results obtained by construction were the following $R=155{\cdot}8$ lbs. and $PAQ=40^\circ\ 5'$; the measurement of the angle was made with a common ivory protractor, so that the number of minutes was determined by judgment: on calculating the parts of the triangle ACD the results obtained were $R=156{\cdot}2$ lbs. and $PAQ=39^\circ\ 48'$. It will be observed that when the construction is made on a small scale and with common instruments we can obtain by the exercise of moderate care a result that can be trusted to within the one hundredth part of the quantity to be determined. The same remark applies to all the questions that were solved by the construction from which the figures at the end of the present volume were copied. If in this example the point A were to be *pushed* along the line A P by a pressure of 120 lbs. the resultant would, of course, be determined by the construction shown in the annexed figure.

Fig. 14.



31. *Condition of equilibrium of three pressures.*—If three pressures, P, Q, and R, whose directions are not parallel, act on a body, it is necessary and sufficient for equilibrium that R be equal and opposite to the resultant of P and Q; the resultant of P and Q being found by the parallelogram of pressures. It is worthy of remark that this condition involves the condition that the directions of the three pressures pass through a common point.

Fig. 15.



Ex. 169.—Three ropes PA, QA, RA are knotted together at the point A; on each a man pulls; the angle $PAQ=120^\circ$, $QAR=132^\circ$ and therefore $RAP=108^\circ$; if the man who pulls on AP exerts a pressure of 24·5 lbs.; find with what pressures the other men must pull that the three may balance each other.

[Produce PA to C and measure off on scale $AC=24\frac{1}{2}$, this line must represent the resultant of Q and R, therefore drawing BC parallel to AQ and CD parallel to AR, the pressures Q and R will be represented by the

lines AD and AB respectively, and can be found by measuring them on scale or by calculating their lengths by trigonometry.

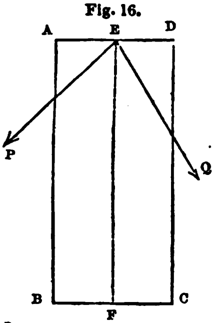
Ans. Q = 31·35 lbs. R = 28.55 lbs.

Ex. 170.—If in the last Example the rope AP were pulled with a pressure of 28 lbs.; AQ with a pressure of 35 lbs.; and AR with a pressure of 12 lbs., determine the angles PAQ, QAR, and RAP.

Ans. QAR = 134°·9′. RAP = 63°·46′. PAQ = 162°·5′.

Ex. 171.—If in Ex. 169 PA is pulled by a pressure of 28 lbs., QA by a pressure of 40 lbs., and the angle PAQ is 135°, determine the magnitude of the pressure along RA, when they are in equilibrium, and the angles RAQ, and RAP. *Ans.* QAR = 135°·34′·30″.
RAP = 89°·25′·30″.
R = 28·28 lbs.

Fig. 16.



Ex. 172.—Let ABCD be a rectangle; AB is 7 ft. long, BC is 3 ft long; join EF the middle points of AD and BC; on E act two pressures, P and Q, in such directions that PEF = 45° and QEF = 60°; the pressure P = 520 lbs; find Q (1) when the resultant of P and Q acts through B, (2) when it acts through F, (3) when it acts through C. *Ans.* (1) 296·9 lbs. (2) 424·6 lbs. (3) 588·5 lbs.

Ex. 173.—A boat is dragged along a stream 50 ft. wide by men on each bank; the length of each rope from its point of attachment to the bank is 72 ft.; each rope is pulled by a pressure of 7 cwts.; the boat moves straight down the middle of the stream; determine the effective pressure in that direction. If, in the next place, one of the ropes is shortened by 10 ft., by how much must the pressure along it be diminished that the direction of the effective pressure on the boat may be unchanged? What will now be the magnitude of the effective pressure?

Ans. (1) 13·13 cwts. (2) $\frac{35}{36}$ cwt. (3) 12·08 cwts.

32. *Cases in which the weight of a body is one of the pressures.*—The following remarks apply to the next and to other questions. (1) It will be proved hereafter that with reference to every heavy body there exists a certain point (called its *centre of gravity*) through which the whole collected weight of the body may be supposed to act in a vertical direction; and that, in the case of any parallelogram, this point is situated at the intersection of its diagonals. (2) In a large number of questions the

solidity of the figures concerned does not enter the question, except so far as it affects the determination of their weight, it being manifest that all the forces act in a single plane; in many such cases a complete enunciation would be long and troublesome to the reader, while an imperfect enunciation is without any real ambiguity; wherever this happens the imperfect enunciation will be preferred; thus, in the next example all the pressures are supposed to act in a vertical plane passing through the centre of gravity; and the diagram ought, strictly speaking, to be that given above, fig. 17, in which the dark lines are all that are shown in the figure which accompanies the example.

Fig. 17.

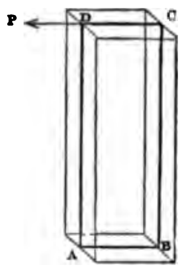
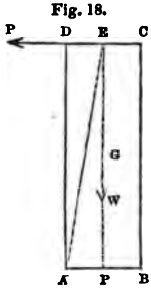


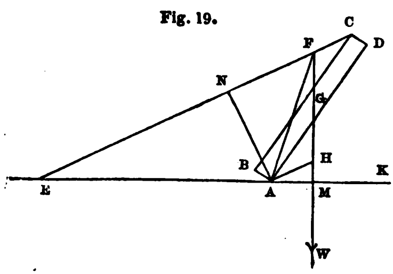
Fig. 18.



Ex. 174.—Let ABCD represent a rectangular mass of oak $2\frac{1}{4}$ ft. thick, AB and AD are respectively 4 ft. and 12 ft. long; it is pulled at D by a horizontal pressure P, and is prevented from sliding by a small obstacle at A; find P when the mass of oak is on the point of turning round A. *Ans.* $1050\frac{3}{4}$ lbs.

[Find G the centre of gravity of ABCD, draw EW vertical meeting DC in E, the weight will act along the line EW, and the resultant of P and W must pass through A since the body is on the point of turning round A;—the remainder of the investigation is conducted as before.]

Fig. 19.



Ex. 175.—ABCD represents a block of oak 35 ft. long and 3 ft. square; the point A is kept from sliding; the mass is held by a rope CE 60 ft. long in such a position that the angle DAK is 57°; determine the direction and amount of the pressure on the point A, and the tension on the string.

[Through G the centre of

gravity of the block draw G W vertical and meeting E C in F; the pressures that balance upon the block are the weight W, the tension T of the string and the resistance of the ground at the point A; this pressure must pass through F, and then we have three pressures acting in known directions through F; &c.]

Ans. (1) Tension 8453 lbs. (2) Pressure on ground 24,592 lbs. making with vertical an angle of 17°39′.

Ex. 176.—On every foot of the length of a wall of brickwork whose section is ABCD a pressure acts on the upper angle C, in a direction making an angle of 45° with the inner side B C; determine this pressure when the resultant of it and of the weight of the wall passes through the angle A at the bottom of the wall; the height of the wall being 20 ft. and its thickness 4 ft. *Ans.* 1584 lbs.

Ex. 177.—If in the last example there were a bracket CE on the inside of the wall, CE being in the same line with D C the top of the wall, and the pressure (inclined at the same angle as before) were applied at E 2 ft. within the wall; what must be its magnitude if the resultant of it and of the weight of one foot of the length of the wall passes through the point A; determine also the point in which the resultant would cut AB the base of the wall if the pressure were the same as in the last example.

Ans. (1) 1810 lbs. (2) $2\frac{2}{3}$ in.

Ex. 178.—If AB are two points in the same horizontal line 10 ft. apart; AC and BC, ropes 10 ft. and 5 ft. long respectively tied by the point C to a weight W of 3 cwts.; determine the tension on each rope.

Ans. Tension on A C = 86·8 lbs. Tension 303·6 lbs.

[The triangle ABC is, of course, fixed in position, the weight W will act vertically through C and be supported by the tensions acting along the ropes.]

Ex. 179.—If AB, BC are two rafters of a roof whose weights are neglected, equally inclined to the vertical at angles of 75°; at the point B is suspended a weight of 5 cwts., find the thrust along each beam.

Ans. 1082 lbs.

[The beams are supposed to be straight lines, *i. e.* their thickness does not enter into the question.]

33. *Triangle of pressures.*—The reader will remark on reference to the figure to Ex. 169, that if lines are drawn parallel to the directions of P, Q, and R respectively, they will form a triangle *abc* similar to A B C, and whose sides will therefore have the same ratios as the pressures, each side being homologous to that pressure to whose direction it is parallel. This fact is frequently of great importance. Thus in Ex. 174, if A E be joined

the sides of the triangle A E F are respectively parallel to the pressures, so that

$$EF : FA :: W : P$$

and since E F, F A, and W are known, P is at once found. Again, in Ex. 175, if A H be drawn parallel to E C, the side of the triangle A F H will be parallel to the pressures, so that

$$FH : HA :: W : T$$
$$\text{and } FH : FA :: W : R$$

from which T the tension on the string, and R the pressure on the ground (or the reaction of the ground to which it is equal and opposite) are at once found.

34. *Parallel pressures.*—If two pressures P and Q act in parallel directions, and towards the same parts at points A and B, their resultant will equal their sum $(P+Q)$ acting in a parallel direction towards the same part, along a line passing through a point X, determined by the following construction:—Join A B, divide A B in X into parts inversely proportional to P and Q, *i. e.* so that

$$AX : XB :: Q : P$$

then X is the point required.

Of course if the body, on which P and Q act, were fixed at the point X, the pressures would balance round that point. It must be understood that a body is said to be fixed by a point, or to have a fixed point, when it can be moved *round* the point, but *not away* from it.

Ex. 180.—If A B is a straight rod 12 ft. long, without weight, and at the end A is hung a weight P of 15 lbs., at the end B a weight Q of 20 lbs., find the point X in the rod round which the weights balance.

Let $AX = x$, then $BX = 12 - x$; hence

$$x : 12 - x :: 20 : 15.$$

$$\therefore x = 6\tfrac{6}{7}, \text{ or } AX = 6\tfrac{6}{7}.\ BX = 5\tfrac{1}{7} \text{ ft.}$$

Ex. 181.—Two men A and B carry a weight of 3 cwts. slung on a pole

the ends of which rest on their shoulders; the distance of the weight from A is 6 ft., and from B 4 ft. Find the pressure sustained by each man.

If P is the pressure sustained by A and Q that sustained by B

$$P+Q=3 \text{ cwts.}$$

$$\text{and } 6:4::Q:P$$

$$\therefore P=1\tfrac{1}{5} \text{ cwt. and } Q=1\tfrac{4}{5} \text{ cwts.}$$

Ex. 182.—There is a beam of oak 30 ft. long and 2 ft. square; at a distance of 1 ft. from one end is hung a weight of 1 ton; how far from that end must the point of support be on which the beam when horizontal will rest, and what will be the pressure on that point?

Ans. (1) 11·61 ft. (2) 9245 lbs.

Ex. 183.—If a mass of granite 30 ft. long, 1 ft. high, and 3 ft. wide is supported in a horizontal position on two points each 3 inches within the ends (and therefore $29\frac{1}{2}$ feet apart), find the pressure on each point of support. *Ans.* 7383 lbs.

Ex. 184.—If in the last Ex. another mass of granite with the same section and half as long is laid lengthwise on the former, their ends being square with each other; determine the single pressure to which their two weights are equivalent, and the line along which it acts, and hence the pressure on the two points of support.

Ans. (1) Resultant equals 22,148 and acts 17·5 feet from one end.

(2) Pressures on point of support respectively 9197 lbs. and 12,950.

Ex. 185.—If in the last case the upper block is shifted round through a right angle in such a manner that middle point of the upper block is exactly over a point in the axis of the lower, and the end of the lower in the same plane with the face of the latter; determine the pressures on the points of support. *Ans.* 7695 lbs. and 14,452 lbs.

Ex. 186.—If in Ex. 180 the rod had weighed 1 lb. per foot find at what distance from A is the point about which the whole will balance.

Ans. 6·64 feet.

[Find the resultant of P and Q as before; and then the resultant of this and of the weight of the rod.]

35. *Principle of the equality of moments.*—The definition of the moment of a pressure is as follows:—If P represent any pressure, and if A be any point, and if on P's direction a perpendicular AN be let fall, and if the number of units of weight in P be multiplied by the number of units of length in AN, the product is called the moment of the pressure P with reference to the point A. The principle of moments in its general form will be found in the second section of this chapter; for present

purposes the following statement will be sufficient. *If any number of pressures acting in the same plane keep a body in equilibrium round a fixed point, and if their moments with reference to that point be taken, the sum of the moments of those pressures which tend to turn the body from right to left round the fixed point, will equal the sum of the moments of those pressures which tend to turn the body from left to right round that point.* This principle affords a very convenient method of solving a very great number of statical questions, as the student will find in the course of the present work; one class of questions may be particularised, it is that class into which enters the consideration of a pressure entirely unknown, except that it passes through a certain point; if we do not want to know this pressure, the principle of moments, *as above stated*, is sufficient for the solution of the question.

The following case will exemplify the mode of applying the principle of moments. In Ex. 175, let it be required only to determine the tension on the rope. Construct the figure to scale (see fig. 19 B); determine G the centre of gravity of the block, draw the vertical line GW, cutting AK in M; draw AN perpendicular to CE; if T is the tension on the rope, and W the weight of the block which can be found to equal 18,388 lbs.; then the moments of T and W are respectively AN × T and AM × 18,388; and the principle of moments assures us that these two are equal. In the construction from which fig. B was drawn, the scale employed was 1 inch to 10 feet; and it was found that AM equals 8·25 ft., and AN equals 18·1 ft.; hence was obtained for T a value of 8381 lbs.; the value of T as determined by calculation is 8453 lbs.

The student is recommended, as an exercise in this important principle, to work by this method all the previous examples in the present chapter, to which it can be readily applied, viz. Ex. 172, 174, 177, 180—186.

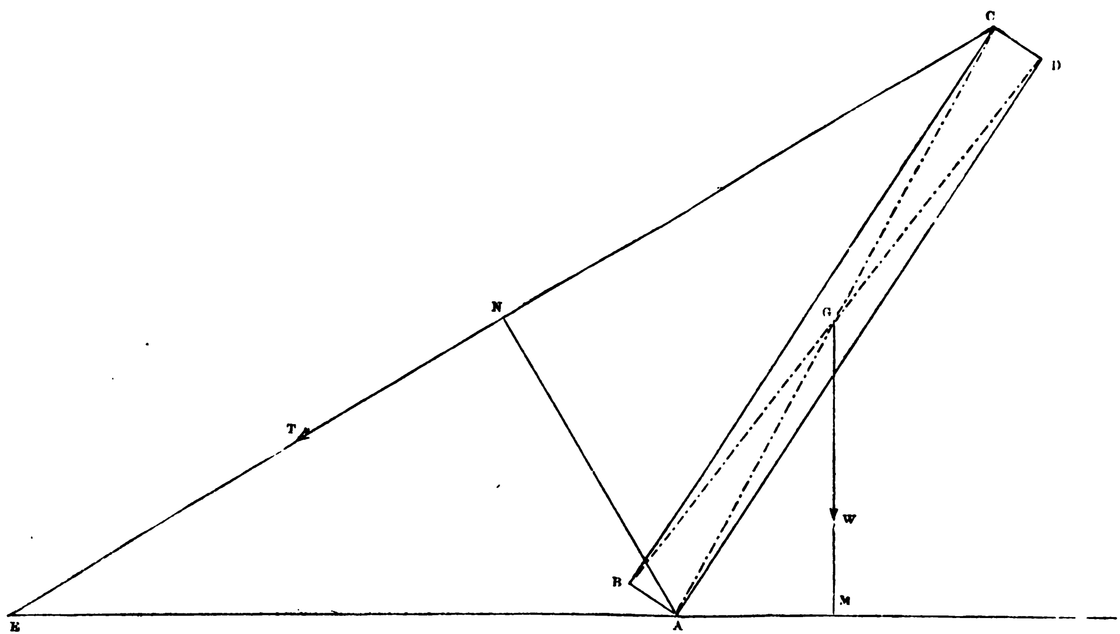


Fig. *b*. p. 88.

SECTION II.

The General Theorems of Statics.

36. *Axioms.*—The following elementary principles, or axioms, are assumed in the demonstrations of the fundamental theorems.

Ax. 1. The line which represents the resultant of two pressures acting on a point, falls within the angle made by the lines that represent those pressures.

Ax. 2. If two *equal* pressures act on a point, the line that represents their resultant bisects the angle between the lines that represent those pressures.

Ax. 3. If a pressure acts upon a body it may be supposed to act indifferently at any point in the line of its direction, provided that point is rigidly connected with the body.

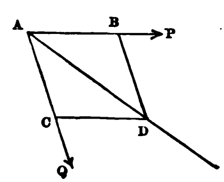
Ax. 4. It is *necessary* and *sufficient* for the equilibrium of any system of pressures, that one of them be equal and opposite to the resultant of all the rest.

Ax. 5. If a system of pressures in equilibrium be imposed on or removed from any system of pressures it will not affect the equilibrium or the resultant of that system.

Proposition 3.

The principle of the parallelogram of pressures is true of the *direction* of the resultant of two equal pressures.

Fig. 20.



Let the pressures P and Q act on the point A along the lines AP and AQ; let AB represent the pressure P, and AC the pressure Q, then will AB equal AC; complete the parallelogram ABCD, and draw the diagonal AD. We are to show that the resultant of P and Q acts along the line AD.

Since AC equals AB it equals CD, therefore the angle CAD is equal to the angle ADC, but since CD is parallel to AB, the angle ADC is equal to the angle BAD, there-

fore the angle BAD equals the angle CAD, and the line AD bisects the angle PAQ; but the direction of the resultant of P and Q bisects the angle PAQ (Ax. 2), therefore AD is the direction of the resultant. Q. E. D.

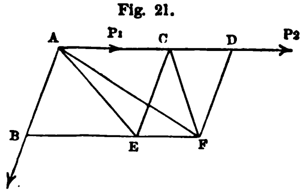
37. *Remark.* — The following proposition may be regarded as the foundation of the whole science of statics; the demonstration generally seems obscure to readers who meet with it for the first time, this results from the somewhat unusual *form* of the proof; it may therefore be well to remark that the demonstration consists of two parts; in the first part it is shown that if the principle is true in two cases, viz. with regard to the pair of pressures P and P_1 and the pair P and P_2, it must also hold good in a third case, viz. in regard to the pair of pressures P and P_1+P_2; this part of the proof is purely hypothetical, as much so as in the case of a demonstration by reduction to an absurdity; the second part of the proof takes up the argument, but as a matter of fact the proposition is true in two certain cases, therefore it must be true in a third case, therefore in a fourth case, and so on.

Proposition 4.

The principle of the parallelogram of pressures is true for the direction of the resultant of any two *commensurable* pressures.

Let the pressure P act on the point A along the line AB, and the pressures P_1 and P_2 on the point A along the line AC; take AB, AC, CD, respectively proportional to P, P_1, and P_2, and complete the parallelograms BC, ED, then is the figure BD a parallelogram; draw the diagonals AE, CF and AF, and suppose the points C, D, E, F, to be rigidly connected with A.

Fig. 21.



(a) The lines AB and AC represent the pressures P and P_1; *assume* that AE is the direction of their resultant; then can P and P_1 be replaced by their resultant acting at A along AE, and, since A and E are rigidly connected, by that resultant acting at E along AE (Ax. 3); but this resultant acting at E can be replaced by its components acting at E, viz. by P_1 along BE, and by P along CE; and these again, since C and F are rigidly connected with E, by P_1 acting at F along BF, and P acting at C along CE.

(b) Since A and C are rigidly connected, P_2 may be supposed to act at C along CD; then CE represents the pressure P, and CD the pressure P_2; *assume* that CF represents the direction of the resultant, then by reasoning in the same manner as in paragraph (a) it can be shown that the pressures P and P_2 can be transferred to F.

(c) Thus it follows from our two assumptions that the pressures P, P_1, P_2 may be supposed to act indifferently on A or F, therefore each of these must be a point in the direction of their resultant, *i. e.* their resultant must act along the line AF. Now AB represents the pressure P and AD the pressure $P_1 + P_2$; hence, if the proposition is true of the pair of pressures P and P_1 and of the pair of pressures, P and P_2, it must also be true of the pair P and $P_1 + P_2$.

(d) But it appears from Prop. 3, that the proposition is true of equal pressures, *i. e.* of the pair p and p, and of the pair p and p, therefore it will be true of the pair p and $p+p$, *i. e.* of p and $2p$; again, since the proposition is true of the pair p and p, and of the pair p and $2p$, it must be true of the pair p and $p+2p$, *i. e.* of p and $3p$; similarly it is true of p and $4p$, of p and $5p$, &c., and generally of p and mp.

(e) Again, since the proposition is true of the pair of pressures mp and p, and the pair mp and p, it must be true of the pair mp and $p+p$, *i. e.* of mp and $2p$, similarly

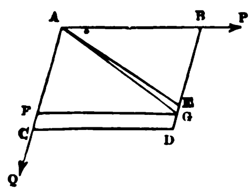
it must be true of mp and $3p$, of mp and $4p$, and generally of mp and np.

(f) Now, any two *commensurable* pressures P and Q must have a common unit (*e. g.* a pound or an ounce, &c.), and therefore can be represented by mp and np; hence the theorem is true of any two commensurable pressures. Q. E. D.

Proposition 5.

The principle of the parallelogram of pressures is true of the direction of the resultant of any two incommensurable pressures.

Fig. 22.



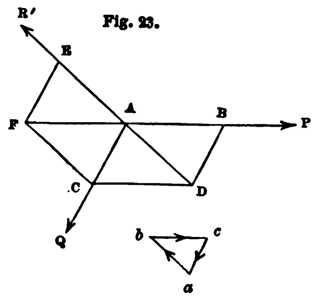
Let P and Q be the two pressures represented by the lines A B and A C, complete the parallelogram A B C D, then will the resultant (R) of P and Q act along the line joining A and D. For if not suppose R to act along any other line, this line must fall within the angle P A Q (Ax. 1), and therefore must cut either C D or D B; let it cut B D in the point E. Now, by continually bisecting A B, a part can be found less than D E, set off distances equal to this part along A C, and let the last of them terminate at F (it cannot terminate at C since A B and A C are incommensurable); therefore F C is less than this part, and therefore also less than D E; draw F G parallel to C D, this line will cut B D, in a point G between D and E, join A G. Suppose A F to represent a pressure Q′ and F C a pressure q, then will Q equal $Q' + q$; now Q′ and P are commensurable, therefore their resultant (R′) will act along the line A G. But the resultant R of P and Q must equal the resultant of P, Q,′ and q; *i.e.* of R′ and q; but R′ acts along A G, and q along A C, and therefore (Ax. 1) their resultant R must act *within* the angle G A Q; but by the supposition it acts along A E *without* the angle G A Q; which is absurd. Therefore, &c. Q. E. D.

Proposition 6.

The principle of the parallelogram of pressures is true of the magnitude of the resultant.

Let P and Q be the two pressures acting on the point A, and let them be represented by the straight lines A B and A C, complete the parallelogram A B C D, and draw the diagonal A D; we have to prove that not only does the resultant R of P and Q act along the line A D, but also that it is represented in magnitude by that line. Suppose R′ to be the pressure which balance P and Q, it must act along D A produced. Let A E represent R′; complete the parallelogram C E, and join A F; the resultant of Q and R′ must act along A F; but since P balances Q and R′, it must act along F A produced; therefore F A B is one straight line, and is parallel to C D, so that F D is a parallelogram. Hence we have F C equal to A D, but F C equals A E, therefore E A equals A D. But R is equal and opposite to R′, which is represented by A E, and therefore R is represented in magnitude by A D. Q. E. D.

Fig. 23.



38. *Application of trigonometry to statics.*—It is manifest that the sides of the triangle A C D are proportional to the three pressures P. Q. R′. which are in equilibrium. .And hence if any triangle *a c d* be drawn similar to A C D, its sides will be proportional to the pressures. Such a triangle will be formed by drawing lines respectively parallel to the directions of the pressure, each pressure being an homologous term to the side parallel to its direction. The pressures as shown by the arrow heads act in the *same* direction round the triangle. A similar

remark applies to a triangle formed by drawing lines perpendicular to the directions of three pressures in equilibrium. The relations between three pressures in equilibrium are thus reduced to the relations between the sides of a triangle; and of course all the trigonometrical relations between the sides and angles of that triangle will be analogous to relations between the pressures and and the angles between their directions. The two of most importance are proved in the following proposition:—

Proposition 7.

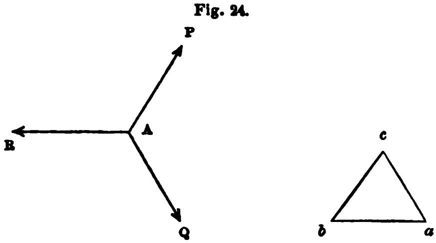
If three pressures, P, Q, R are in equilibrium, and act upon a point A, to show that the following relations obtain:—

(1) $\quad P : Q :: \sin QAR : \sin RAP.$

$\quad\quad Q : R :: \sin RAP : \sin PAQ.$

(2) $\quad R^2 = P^2 + Q^2 + 2\,P\,Q \cos PAQ.$

Fig. 24.



(1) Draw the triangle $a\,b\,c$ whose sides bc, ca, ab, are respectively parallel to the pressures P, Q, R. Then it is evident that the angles a, b, c are respectively equal to $180° - QAR$, $180° - RAP$, $180° - PAQ$; now

$$bc : ca :: \sin bac : \sin cba :: \sin QAR : \sin RAP$$
$$ca : ab :: \sin cba : \sin acb :: \sin RAP : \sin PAQ$$

But by (Art. 38)

$$bc : ca :: P : Q$$
$$ca : ab :: Q : R$$
$$\therefore P : Q :: \sin QAR : \sin RAP$$
$$\text{and } Q : R :: \sin RAP : \sin PAQ.$$

These proportions are sometimes expressed by the rule,

"If three pressures are in equilibrium, each pressure is proportional to the sine of the angle contained by the other two."

(2) Employing the same figure, we have, by a well-known theorem in trigonometry,

$$ab^2 = bc^2 + ca^2 - 2\ bc.\ ca. \cos bca.$$

Now bca is the supplement of PAQ, so that cos PAQ= − cos bca.

$$\therefore ab^2 = bc^2 + ca^2 + 2\ bc.\ ca. \cos PAQ.$$

But bc, ca, ab, are respectively proportional to the pressures P, Q, R.

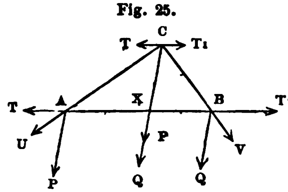
$$\therefore R^2 = P^2 + Q^2 + 2PQ \cos PAQ. \qquad \text{Q. E. D.}$$

Proposition 8.

To determine the resultant of two pressures acting in parallel directions and towards the same parts.

Let P and Q be the pressures acting respectively on the points A and B; join AB; suppose any two equal and opposite pressures T, T_1 to act at A and B respectively along the line AB; these pressures being separately in equilibrium will not affect the resultant of P and Q (Ax. 5), therefore the required resultant will be that of

Fig. 25.



T, P, Q, and T_1, *i. e.* of U and V, if U is the resultant of T and P, and V the resultant of Q and T_1. But since U's direction lies within the angle TAP and V's within the angle QBT_1, their directions will meet when produced; let them be produced and meet in C; then if C is rigidly connected with the body, U and V may be supposed to act at C; through C draw CX parallel to AP or BQ; but U acting at C can be resolved into P, acting along CX, and T, acting parallel to BA, and similarly V can be resolved into Q acting along CX, and T_1 acting

parallel to AB; hence the required resultant will be that of T, T_1, P, and Q acting at C; or, since T and T_1 are in equilibrium, of P and Q acting along CX, *i. e.* the resultant is P+Q acting at X along a line parallel to AP or BQ, and towards the same part.

Next, to find X. Since U is the resultant of P and T, those pressures will be proportional to the sides of the triangle AXC.

$$\therefore AX : XC :: T : P.$$
$$\text{Similarly } CX : XB :: Q : T_1.$$
$$\therefore (\text{Ex æq.})\ AX : XB :: Q : P.$$

Or the point X divides AC in the inverse ratio of the pressures, which is the proof of the rule already given (Art. 34). Q. E. D.

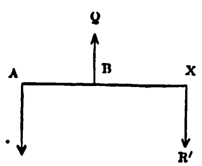
Cor. 1.— Hence we immediately deduce the conditions of the equilibrium of three parallel pressures. For let P, Q, and R be the pressures of which we will suppose that P and Q act towards the same parts, then R must be equal and opposite to their resultant (Ax. 4); hence—

(1.) R must equal P+Q acting towards the opposite part.

(2.) R's direction must cut the line joining the points of application of P and Q into parts inversely proportional to those pressures.

Cor. 2.—Hence, also, we can determine the resultant of two parallel pressures acting towards contrary parts. Thus suppose P acting at A and Q acting at B to be the pressures, of which let Q be the greater; now, if R′ is the pressure that balances P and Q, it must be equal and opposite to their resultant R; but R′+P=Q, and AB : BX :: R′ : P, *i. e.* AB+BX : BX :: R′+P : P, or AX : BX :: Q : P.

Fig. 26.



i. e. the resultant equals Q−P, and acts towards the same part as Q at a point X, whose distances from A and B are inversely as the pressures, and so taken that the greater pressure acts between the resultant and the lesser pressure.

39. *Statical Couples.*—The equations which determine R′, *i. e.* the pressure that balances two parallel pressures acting towards contrary parts, are—

$$R' = Q - P;$$

and $$AB : BX :: R' : P;$$

or $$BX = \frac{AB \cdot P}{R'}$$

Now, if P is very nearly equal to Q, the pressure R′ must be very small, and since AB·P has a fixed value, BX must be very large; and if we consider the limiting case, when P=Q, we have R′ indefinitely small and BX indefinitely great; that is to say, two equal pressures acting in parallel directions towards contrary parts cannot be brought to rest by any single pressure; or, which is the same thing, two such pressures do not have a single resultant; they constitute what is called a *statical couple.*

40. *The use of the positive and negative sign to denote the direction of a pressure.*—Since a line can be taken to represent a pressure, and since if $+a$ is used to denote a line of a feet (or other units), measured to the right from a fixed point, then $-a$ must be used to denote a line of a feet measured to the left from that point, it should seem that the same principle ought to be applicable to pressures, and that if +P denote a pressure of P lbs. acting to the right along a given line, then − P must denote a pressure of P lbs. acting towards the left along that line. That the principle so commonly used in geometry is correctly applied to pressures, will be evident from a little considera-

tion. Thus, if P and Q are two pressures acting to the right along a line, and R their resultant, we have

$$R = P + Q. \qquad (1)$$

If Q act to the left and be less than P, R will act to the right, and we have

$$R = P - Q. \qquad (2)$$

If, however, Q be greater than P, R will act to the left, and we have

$$R = Q - P. \qquad (3)$$

Here we have three equations to express a certain result; but if we suppose $P + Q$ to be an *algebraical* sum, these three equations can be included in one, viz.

$$R = P + Q. \qquad (4)$$

It is quite plain the (4) includes (1) and (2), it also includes (3), since that equation can be written

$$-R = P - Q.$$

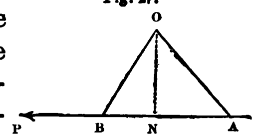
This circumstance enables us to abbreviate considerably the enunciation of the general theorems of statics. Thus: —(a) *If any number of pressures act along a line their algebraical sum equals their resultant;* a statement which is identical in meaning with the following (b)—*If any number of pressures act along a line their resultant will equal their sum, if they all act in one direction; but if not, the sum of those that act in one direction must be found, and also the sum of those that act in the other direction, and then the resultant will equal the difference between these sums, and will act in the direction of the greater.* Of course the condition of equilibrium of any number of pressures acting along a line will be that their algebraical sum equals zero.

The same principle can be applied to the moments of pressures. If we measure the moment of a pressure with reference to a certain point, it is usual to reckon it positive

if it tend to turn the body round that point in a direction contrary to that in which the hands of a watch move, or when it tends to turn the body from right to left upwards. If this assumption be made, then the moment of any other pressure must be reckoned negative which tends to turn the body in the contrary direction round that point. It will be remarked that in the figure belonging to the first section of the following proposition (9) the moments of P, Q, R, with respect to O are positive; in the figure belonging to the second section the moments of Q and R are positive, and of P negative.

41. *Representation of a moment by an area.*—Let the line AB represent a pressure P, and from a point O let fall a perpendicular ON on AB or AB produced, join OA, OB; then twice the area of the triangle AOB equals the product of ON and AB, *i.e.* the product of the perpendicular on P's direction and the line that represents P; hence twice the area of the triangle AOB represents the moment of the pressure P with respect to the point O.

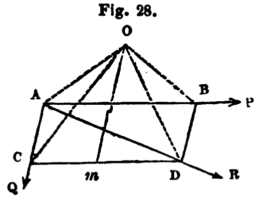
Fig. 27.



Proposition 9.

The moment, with respect to any point, of the resultant of any two pressures whose directions are not parallel is equal to the algebraical sum of the moments of those pressures with respect to that point.

Fig. 28.



Let P and Q be the pressures acting on the point A; let AB represent P, and AC represent Q; complete the parallelogram ABDC, and draw the diagonal AD, then AD represents the resultant R.

(1.) Let the point O about which the moments are to be measured fall beyond AB, as shown in the annexed figure; in this case all the moments are positive; we have, therefore, to show that

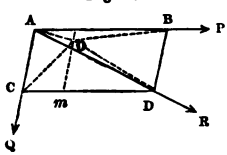
$$M^tR = M^tP + M^tQ.$$

Join OA, OB, OC, OD. Then we have

$$\begin{aligned} M^tR - M^tP &= 2\triangle AOD - 2\triangle AOB \\ &= 2\triangle ABD - 2\triangle OBD \\ &= BC - Bm = Am = 2\triangle AOC \\ &= M^tQ \\ M^tR &= M^tP + M^tQ \end{aligned}$$

(2.) Let the point O fall on the inside of the parallelogram BC, and within the angle PAR as shown in the annexed figure; in this case the moments of Q and R with respect to O are positive, and that of P negative, so that we have to prove that

Fig. 29.



$$M^tR = M^tQ - M^tP.$$

Make the same construction as before.

Then we have

$$\begin{aligned} M^tR + M^tP &= 2\triangle AOD + 2\triangle AOB \\ &= 2\triangle ABD - 2\triangle OBD \\ &= BC - Bm = Am = 2\triangle AOC = M^tQ \\ \therefore M^tR &= M^tQ - M^tP \end{aligned}$$

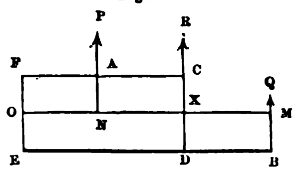
It will be found that a similar proof applies to any other position of O. Hence, &c. Q. E. D.

Proposition 10.

The moment, with reference to any point, of the result-

ant of two parallel pressures is equal to the algebraical sum of the moments of those pressures with reference to that point.

Fig. 30.



Let P and Q be the two pressures, acting towards the same parts; O the point about which the moments are measured; draw OMN perpendicular to the directions of the pressures; divide NM in X, so that NX : XM :: Q : P, then the resultant R will pass through X in a direction parallel to NP or MQ, and will equal P+Q; let AN and MB represent P and Q on scale; complete the rectangles FN and EM; let EB and FA produced cut R's direction in C and D respectively, then CD represents R scale. Since all the moments are positive, we have to prove that

$$M^tR = M^tP + M^tQ$$

or, which is the same thing, we have to show that

$$EC = OA + OB$$

Now

$$NX : XM :: Q : P$$

or

$$NX : XM :: XD : XC$$

But the angle NXC = DXM (Eucl. 15—I.); therefore (Eucl. 16—VI.) rectangle NC = rectangle DM.

But $EC = OA + NC + OD = OA + MD + OD = OA + OB$

$$\therefore M^tR = M^tP + M^tQ$$

A similar proof will apply to every position of O. Hence, &c. Q. E. D.

Cor.—A *complete* proof of Propositions 8 and 9 can only result in strictness from a discussion of *every* particular case; this, however, is not nécessary, as the student can easily adapt the proofs to other cases; for instance, in

Prop. 10 the point O might be taken within NM; or P and Q might act towards contrary parts: in Prop. 9 the point O might fall so that OM would cut CD *produced*, or might fall within the angle QAR, &c.

42. *Extension of the principle of moments to any number of pressures.*—If we have any number of pressures P_1, P_2, P_3, . . . P_n, and if we first find the resultant R_1 of any two P_1 and P_2, next the resultant R_2 of R_1 and P_3, then R_3 the resultant of R_2 and P_4; and if we can continue this process up to P_n we shall obtain the resultant R of the whole system; a little consideration will render it manifest that any system of pressures acting in one plane can be thus reduced to a single pressure, with one exception, viz. when the system can be reduced to a couple; this exception is excluded from the following reasoning. Now, if we take any point O, and take with reference to it the moments of the pressures, we shall have

$$\begin{aligned} M^tR_1 &= M^tP_1 + M^tP_2 \\ M^tR_2 &= M^tR_1 + M^tP_3 \\ M^tR_3 &= M^tR_2 + M^tP_4 \\ &\vdots \\ M^tR &= M^tR_{n-2} + M^tP_n \end{aligned}$$

$$\therefore \text{by addition } M^tR = M^tP_1 + M^tP_2 + M^tP_3 + M^tP_4 + .. + M^tP_n,$$

the sum on the right hand side of the equation being an algebraical sum. Hence *if any pressures act in a plane the moment of their resultant with reference to any point is equal to the algebraical sum of the moments of the pressures with respect to that point,* which is the principle of moments in its most general form.

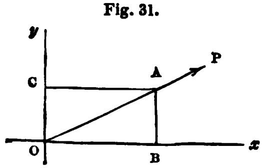
Of course if the point be taken in the direction of the resultant, its moment, and therefore the algebraical sum of the moments of the pressures, will equal zero. Now, if a body acted on by any pressures be kept at rest round

a fixed point, the resultant must pass through that point; and therefore in this case the algebraical sum of the moments of the pressures round that point will equal zero; a statement which coincides with that already given. (Art. 35.)

43. *The transfer of pressures in parallel directions.*—Suppose a pressure P to act along a certain line, then if it be made to act towards the same part along a parallel line, it is said to be transferred in a parallel direction. It is evident from Propositions 4 and 8 that any two pressures can be transferred in parallel directions to any point in the direction of their resultant without changing their effect; and the same will be true of any number of pressures: thus if P_1, P_2, P_3 be the pressures, find R_1 the resultant of P_1 and P_2, and R the resultant of R_1 and P_3. Now, we may conceive R to act at any point O of its direction, and there we can resolve it into R_1 and P_3 acting parallel to their former directions, and R_1 into P_1 and P_2 acting parallel to their former directions; and thus we have transferred in parallel directions all three pressures to a point O in the direction of their resultant. The same reasoning will manifestly hold good of any number of pressures.

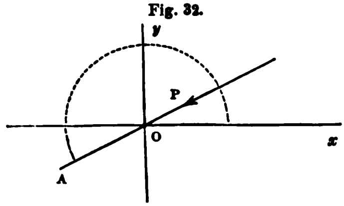
44. *The rectangular components of a pressure.*—Let Ox, Oy be two rectangular axes, and let P be a pressure acting on O along the line OP; let OA be the line which represents the pressure P, and let the angle it makes with the axis of x, viz. xOA, equal θ; now if the parallelogram OBAC be completed, P will be equivalent to two pressures respectively represented by OB, and OC, and since these pressures are at right angles to one another, they are called the rectangular components of P with respect to the axes Ox and Oy; again, since OC=

Fig. 31.



OA sin θ and OB = OA cos θ, it is plain that the rectangular components of P are P cos θ along the axis O x and P sin θ along the axis of y. It will be remarked that if we always measure θ in the same direction viz. upwards from O x, that P cos θ, and P sin θ give not only the *magnitude* but also the *directions* in which the components act: — thus if we suppose P to act *towards* O, the line which represents the pressure is OA, so that θ is not xOP, but x OA, indicated by the dotted arc; and then, since θ lies between 180° and 270°, both P sin θ, and P cos θ will be negative, as they ought to be.

Fig. 32.



Proposition 11.

To determine the resultant of any number of pressures acting on a point; and to infer the conditions of equilibrium of a system of pressures acting on a point.

(a) Let P_1, P_2, P_3,.... be the pressures acting on any given point O, through O draw two rectangular axes x and y, and let θ_1, θ_2, θ_3,... be the angles that the lines representing the pressure make with the axis of x. Then these pressures can be replaced by their rectangular components along the axes of x and y, *i. e.* by

$P_1 \cos \theta_1, P_2 \cos \theta_2, P_3 \cos \theta_3,\ldots$ along the axis of x, and by
$P_1 \sin \theta_1, P_2 \sin \theta_2, P_3 \cos \theta_3,\ldots$ along the axis of y.

Now the former set is equivalent to a single pressure X acting along the axis of x, and the latter to a single pressure Y acting along the axis of y, provided

$$X=P_1 \cos \theta_1+P_2 \cos \theta_2+P_3 \cos \theta_3+\ldots.$$
$$Y=P_2 \sin \theta_2+P_2 \sin \theta_2+P_3 \sin \theta_3+\ldots.$$

Now, if R be the resultant of X and Y, and ϕ the angle which the line representing it makes with Ox, we must have

$$R \cos \phi = X \qquad (1)$$
$$R \sin \phi = Y \qquad (2)$$

which equations determine R and ϕ. It will be remarked the determination is free from ambiguity, since the signs of X and Y will give the signs of $\cos \phi$ and $\sin \phi$, and therefore determine the *quadrant* in which the line representing R falls. Of course the magnitudes of R and ϕ are given by the equation

$$R^2 = X^2 + Y^2 \qquad (3)$$

$$\text{and } \tan \phi = \frac{Y}{X} \qquad (4)$$

(b) To obtain the conditions of equilibrium of $P_1, P_2, P_3, \ldots$

It must be remembered that it is necessary and sufficient for the equilibrium of these pressures that P_1 be equal and opposite to the resultant of $P_2, P_3 \ldots$ (Ax. 4), so that the rectangular components of this resultant must be $-P_1 \sin \theta_1$ and $-P_1 \cos \theta_1$, therefore the required conditions are

$$-P_1 \sin \theta_1 = P_2 \sin \theta_2 + P_3 \sin \theta_3 + \ldots.$$
$$\text{and } -P_1 \cos \theta_1 = P_2 \cos \theta_2 + P_3 \cos \theta_3 + \ldots.$$
$$\text{or } \quad P_1 \sin \theta_1 + P_2 \sin \theta_2 + P_3 \sin \theta_3 + \ldots. = 0$$
$$\text{and } \quad P_1 \cos \theta_1 + P_2 \cos \theta_2 + P_3 \cos \theta_3 + \ldots. = 0$$

That is to say—"It is necessary and sufficient for the equilibrium of any system of pressures acting on a point, that the sums of their components along each of two rectangular axes be separately zero."

Proposition 12.

To determine the resultant of any system of pressures acting in a plane.

Take Ox, and Oy any two rectangular axes, and let $P_1, P_2, P_3, \ldots.$ be the pressures, and $\theta_1, \theta_2, \theta_3, \ldots$ the

angles which their directions make with the axis of x — or rather which the lines that would represent them if they were transferred to the origin make with the axis of x.

(a.) Suppose the pressures to be transferred in parallel directions to the origin, then if R be the resultant of the transferred pressures and ϕ the angle it makes with the axes of x, R and ϕ are completely determined by the equations,

$$R \sin \phi = P_1 \sin \theta_1 + P_2 \sin \theta_2 + P_3 \sin \theta_3 + \ldots$$
$$R \cos \phi = P_1 \cos \theta_1 + P_2 \cos \theta_2 + P_3 \cos \theta_3 + \ldots$$

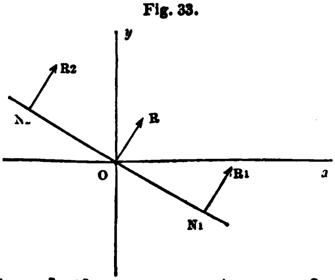
(b.) Since the pressures $P_1, P_2, P_3, \ldots$ can be transferred in parallel directions to any point in their resultant without changing their effect (Art. 43), it follows that the resultant must be equal and parallel to R, and act towards the same part.

(c.) If we determine the moments of the pressures with reference to the origin O, and if p be the perpendicular on R's direction from O, we shall have, by the principle of moments,

$$R\,p = M^t P_1 + M^t P_2 + M^t P_3 + \ldots. \quad (3)$$

Now, as we know the magnitude of R, this equation gives the magnitude of p.

Fig. 33.



(d.) Hence, O being the origin, and OR the direction of R, if we draw through O a perpendicular to R, and measure ON_1 and ON_2 each equal to p, and suppose pressures equal and parallel to R to act through N_1 and N_2 and towards the same part, one of these two must be the required resultant; to ascertain which, we must observe that equa-

tion (3) gives the sign of the moment of the resultant; hence the required resultant will be R_1, if its moment with respect to O has the required sign, if not R_2 will be the resultant. For instance, if R acted as shown in the diagram, and if $M^t\ P_1 + M^t\ P_2 + M^t\ P_3 + \ldots.$ were negative, R_2 would be the required resultant.

Cor.—It will be remarked that the above determination is entirely free from ambiguity. It may also be remarked that if the case arise in which

$$P_1 \sin\theta_1 + P_2 \sin\theta_2 + P_3 \sin\theta_3 + \ldots = 0$$
$$P_1 \cos\theta_1 + P_2 \cos\theta_2 + P_3 \cos\theta_3 + \ldots = 0$$
$$M^t\ P_1 + M^t\ P_2 + M^t\ P_3 + \ldots = \text{some finite value,}$$

the system of pressures will reduce to a couple.

Proposition 13.

To determine the conditions of equilibrium of any number of pressures acting in a plane.

Let $P_1, P_2 P_3, P_4, \ldots$ be the pressures and $\theta_1, \theta_2, \theta_3, \theta_4, \ldots$ the angles their directions make with the axis of x—as already explained. Now, if R be the resultant of $P_2, P_3, P_4, \ldots.$ we have for determining R,

$$R \sin\phi = P_2 \sin\theta_2 + P_3 \sin\theta_3 + P_4 \sin\theta_4 + \ldots.$$
$$R \cos\phi = P_2 \cos\theta_2 + P_3 \cos\theta_3 + P_4 \cos\theta_4 + \ldots.$$
$$\text{and } M^t\ R = M^t\ P_1 + M^t\ P_3 + M^t\ P_4 + \ldots..$$

But if the required pressures are in equilibrium it is necessary and sufficient that R be equal and opposite to P_1 (Ax. 5) *i. e.* $R \sin\phi = -P_1 \sin\theta_1$, $R \cos\phi = -P_1 \cos\theta_1$, and $M^tR = -M^tP_1$. Therefore the required conditions are,

$$-P_1 \cos\theta_1 = P_2 \sin\theta_2 + P_3 \sin\theta_3 + P_4 \sin\theta_4 + \ldots$$
$$-P_1 \cos\theta_1 = P_2 \cos\theta_2 + P_2 \cos\theta_3 + P_4' \cos\theta_4 + \ldots$$
$$-M^t\ P_1 = M^tP_2 + M^t\ P_3 + M^t\ P_4 + \ldots.$$

or the conditions necessary and sufficient for equilibrium are,

$$\begin{aligned} P_1 \sin\theta_1 + P_2 \sin\theta_2 + P_3 \sin\theta_3 + \dots &= 0 \\ P_1 \cos\theta_1 + P_2 \cos\theta_2 + P_3 \cos\theta_3 + \dots &= 0 \\ M^t\, P_1 + M^t\, P_2 + M^t\, P_3 + \dots &= 0. \end{aligned}$$

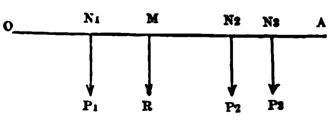
That is to say, "It is necessary and sufficient for the equilibrium of any system of pressures acting in one plane that the sums of the components taken with respect to any two lines at right angles to each other be separately equal to zero, and also that the sums of the moments of the pressures with respect to any one point be zero."

Proposition 14.

To determine the resultant of any number of parallel pressures.

Let P_1, P_2, P_3, ... be the pressures; take any point O and let fall from it the line OA perpendicular to the directions of the pressures, and cutting them in N_1, N_2, N_3, ... let $ON_1 = p_1$, $ON_2 = p_2$, $ON_3 = p_3$ ...; also let R be the resultant of the pressures, and let its direction cut the line OA in M, and let $OM = r$; we have to find the magnitudes of R and r.

Fig. 34.



Now, the pressures may be transferred in parallel directions to any point in the direction of their resultant: if this be done they will all act along the same line, and therefore their resultant must equal their sum, or

$$R = P_1 + P_2 + P_3 + \dots.$$

again, the moment of R round O must equal the sum of the moments of the separate pressures, therefore

$$R r = P_1 p_1 + P_2 p_2 + P_3 p_3 + \dots$$

The former equation gives R and the latter r.

Cor.—It will be remarked, that if we suppose the directions of all the pressures to be turned through a certain angle x, but still to pass through the same points N_1, N_2, N_3,... that the direction R will still pass through M; for let p'_1 p'_2 p'_3... r' be the perpendiculars on the directions of the pressures; when this has been done, then

$$R r' = P_1 p'_1 + P_2 p'_2 + P_3 p'_3 + \dots.$$

now, $$p'_1 = p_1 \cos x, p'_2 = p_2 \cos x, p'_3 = p_3 \cos x, \dots$$

$$\therefore \frac{R r'}{\cos x} = P_1 p_1 + P_2 p_2 + P_3 p_3 + \dots.$$

and $\therefore$ $$\frac{r'}{\cos x} = r$$

But the new direction of R must cut the line OA in a point whose distance from O is $\frac{r'}{\cos x}$, and therefore it will still pass through the point M. In other words, the position of the system of parallel pressures may be changed in any manner we please, provided the relative distances of N_1, N_2, N_3, ... are unchanged and the pressures continue to act in parallel directions through those points, and still the resultant will pass through M, which is therefore called the centre of parallel pressures. It will appear from the next proposition that there is a centre of parallel pressures for every system of parallel pressures, whether the points of applications of the pressures be in the same straight line or not.

Cor. 2.—If we suppose the pressures P_1, P_2, P_3,.... to be the weights of heavy points, the centre of parallel pressures will be the centre of gravity of those points; hence the position of the centre of gravity of a number of weights arranged along a line will be given by the above equation, viz.

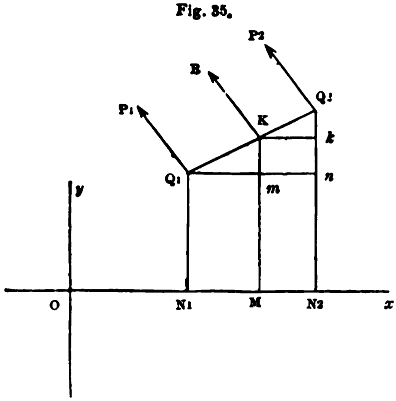
$$r(P_1 + P_2 + P_3 + \dots) = P_1 p_1 + P_2 p_2 + P_3 p_3 + \dots$$

Proposition 15.

To determine the centre of parallel pressures of any system of such pressures acting on a plane.

(1.) Consider the case of two parallel pressures P_1, P_2; let them act at the points Q_1, Q_2, the co-ordinates of which are $ON_1 = x_1$ $N_1Q_1 = y_1$, $ON_2 = x_2$, $N_2Q_2 = y_2$. Divide $Q_1\,Q_2$ in K, so that

Fig. 35.



$$Q_1K : KQ_2 :: P_2 : P_1$$

then the resultant R_1 of P_1 and P_2 will equal $P_1 + P_2$, and will act at K; let the co-ordinates of K be $OM = \bar{x}_1$ and $KM = \bar{y}_1$; through Q_1 and K draw lines parallel to $O\,x$, then by Eucl. (2—VI.) we have

$$Q_1K : KQ_2 :: Q_1\,m : m\,n :: \bar{x}_1 - x_1 : x_2 - \bar{x}_1.$$
$$\therefore \bar{x}_1 - x_1 : x_2 - \bar{x}_1 :: P_2 : P_1.$$
$$\therefore P_1\,\bar{x}_1 - P_1\,x_1 = P_2\,x_2 - P_2\,\bar{x}_1.$$
$$\text{or, } \bar{x}_1(P_1 + P_2) = P_1\,x_1 + P_2\,x_2.$$

Again, since $Q_1K : KQ_2 :: K\,m : Q_2\,k$, we shall obtain, by reasoning in a precisely similar manner, that

$$\bar{y}_1(P_1 + P_2) = P_1\,y_1 + P_2\,y_2.$$

(2.) Suppose there are three pressures, P_1, P_2, P_3; find R_1 the resultant of P_1 and P_2, acting at the point $\bar{x}_1\,\bar{y}_1$, then we have

$$R_1 = P_1 + P_2. \qquad (1)$$
$$\bar{x}_1(P_1 + P_2) = P_1\,x_1 + P_2\,x_2. \qquad (2)$$
$$\text{and } \bar{y}_1(P_1 + P_2) = P_1\,y_1 + P_2\,y_2. \qquad (3)$$

Find R the resultant of R_1 and P_3, acting at the point $\bar{x}\,\bar{y}$, then we have

$$R = R_1 + P_3 = P_1 + P_2 + P_3.$$

$$\bar{x}(R_1 + P_3) = R_1\bar{x}_1 + P_3 x_3.$$

or, $$\bar{x}(P_1 + P_2 + P_3) = (P_1 + P_2)\,\bar{x}_1 + P_3\,x_3, \qquad (4)$$

and $$\bar{y}(R_1 + P_3) = R_1\,\bar{y}_1 + P_3\,y_3,$$

or, $$\bar{y}(P_1 + P_2 + P_3) = (P_1 + P_2)\,\bar{y}_1 + P_3\,y_3. \qquad (5)$$

Hence, adding together (2) and (4), and also (3) and (5), we obtain

$$\bar{x}(P_1 + P_2 + P_3) = P_1\,x_1 + P_2\,x_2 + P_3\,x_3. \qquad (6)$$

$$\bar{y}(P_1 + P_2 + P_3) = P_1\,y_1 + P_2\,y_2 + P_3\,y_3. \qquad (7)$$

The same proof can evidently be extended to four, five, or any number of pressures. Q. E. D.

Cor. 1.—If the points of application of the pressures had been situated in space of three dimensions, and referred to three co-ordinate planes, a precisely similar proof would have given us

$$\bar{x}(P_1 + P_2 + P_3 + \ldots) = P_1 x_1 + P_2 x_2 + P_3 x_3 + \ldots$$

$$\bar{y}(P_1 + P_2 + P_3 + \ldots) = P_1 y_1 + P_2 y_2 + P_3 y_3 + \ldots$$

$$\bar{z}(P_1 + P_2 + P_3 + \ldots) = P_1 z_1 + P_2 z_2 + P_3 z_3 + \ldots.$$

The point determined by the coordinates $\bar{x}\,\bar{y}\,\bar{z}$ is called the *centre of parallel pressures.* It will be remarked that precisely the same values would be obtained in whatever order the pressures had been taken, consequently a system of parallel pressures has only one centre. It, of course, follows from this that no body or system of bodies can have more than one centre of gravity.

Cor. 2.—If the case should arise in which

$$P_1 + P_2 + P_3 + \ldots\ldots = 0,$$

$$\text{but } P_1\,x_1 + P_2\,x_2 + P_3\,x_3 + \ldots = A$$

G

where A denotes some determinate value, the system reduces to a couple; and in this case there is *no* centre of parallel pressures in finite space. If the pressures are the weights of parts of the body they act towards the same parts, and therefore their sum can never be zero, so that every body and system of bodies must have one, and only one centre of gravity, which can be determined by the above equations.

CHAP. IV.

OF THE CENTRE OF GRAVITY.

SECTION I.

45. *Definition of the centre of gravity.*—It has been already remarked that the weight of a body is an instance of a distributed pressure, and that it can be treated as a single pressure by supposing it to be collected at a certain point, called its centre of gravity. The formal definition of the centre of gravity is as follows:—*The centre of gravity of a body or system of bodies is that point at which we may suppose their whole weight to act without changing its statical effect.* That, as a matter of fact, every body has a centre of gravity, is shown in the corollary to Proposition 15. In determining the centre of gravity of any figure, it is assumed that a heavy line can be supposed to be made up of heavy points, a heavy plane of heavy parallel lines, and a solid of heavy parallel planes. It is also assumed that every figure is of uniform density, unless the contrary is specified.

Ex. 187.—Determine the centre of gravity of a uniform straight line AB.

The line AB may be conceived to be made up of a number of equally heavy points distributed uniformly along it (like beads on a wire); now if we take the two extreme points, the resultant of their weights will pass through the middle point of AB, that of the weights of the next two will pass through the middle point of AB, and in like manner that of each successive pair; consequently the weight of the whole will act through the middle point of AB, which is therefore the centre of gravity of the whole, or of the heavy line AB.

46. *Method of determining the centre of gravity of a plane area.*—Let ABCD be the plane area; we may conceive it to be made up of a set of parallel heavy lines, such as BD, EF. . . drawn in any direction. If we can find a set of parallel lines which are all bisected by a single line AC, the centre of gravity of each line must be in AC, and therefore that of the whole figure must be in AC. If, moreover, we can determine a second line bisecting another set of parallel lines, we know that the centre of gravity must also be in this second line, and must therefore be at its point of intersection with AC. By this method, the centres of gravity of many simple figures can be determined: it also suggests a practical means of determining the centre of gravity of any plane area whatever. Suppose the figure to be cut out carefully to the required shape in card-board or tin; suppose it to be suspended by a fine thread from any point B; now the pressures in equilibrium are the tension of the string and the weight of the body; they must therefore act along the same line, so that the required centre of gravity must be in the prolongation BC of AB; this prolongation can easily be marked by suspending a plumb-line from A. Again, suspend the body by a fine thread DE fastened to any other point E, and draw the prolongation of this line, viz. EF; the centre of gravity must be in EF, and therefore at G, the point of intersection of EF and BC.

Fig. 36.

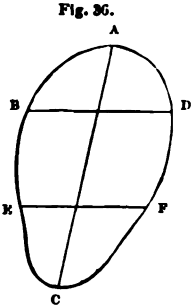
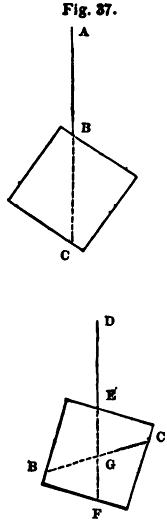


Fig. 37.



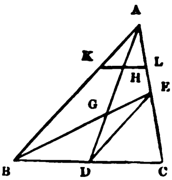
Ex. 188.—Show that the centre of gravity of the area of a circle is at its centre.

Since any diameter bisects all lines in the circle drawn perpendicularly to it, the centre of gravity must be in *any* diameter, and therefore in the centre of the circle.

Ex. 189.—Show that the centre of gravity of an ellipse must be at its centre.

Ex. 190 - Determine the centre of gravity of a triangle.

Fig. 38.



Let ABC be any triangle, bisect BC in D and join AD; draw any line KL parallel to BC cutting AD in H; then by similar triangles we have

$$KH : HA :: BD : DA$$
$$HA : HL :: DA : DC$$
$$\therefore \text{(Ex. aeq.)} \quad KH : HL :: BD : DC.$$

But BD is equal to DC, therefore KH is equal to HL, or KL is bisected by AD; and the same being true of any line drawn parallel to BC, the centre of gravity of the triangle must be in AD. Again, if AC is bisected in E and BE is drawn, the centre of gravity will be in BE, and therefore must be at G, the point of intersection of AD and BE.

It can be easily proved that $GD = \frac{1}{3}AD$. For join ED, then because $AE = EC$, and $BD = DC$ we have

$$AE : EC :: BD : DC,$$

and therefore ED is parallel to AB; hence the triangle DEG is similar to ABG and EDC to ABC;

$$\therefore DG : DE :: GA : AB$$

and $$DE : DC :: AB : BC$$

$$\therefore \text{(Ex aequali)} \; DG : DC :: GA : BC$$

But $$DC = \tfrac{1}{2}BC \therefore DG = \tfrac{1}{2}GA = \tfrac{1}{3}DA.$$

Ex. 191.—Show that the centre of gravity of a parallelogram is at the intersection of the diagonals.

47. *Centre of gravity of solids.*—The above method can easily be extended to the case of solids; we may suppose them to be made up of heavy parallel planes: if we can show that the centres of gravity of these all lie along a line, we know that the centre of gravity of the solid must be in that line, and if two such lines can be found, the centre of gravity of the solid must be at their point of intersection.

Ex. 192.—Show that the centre of gravity of a sphere is at its centre.

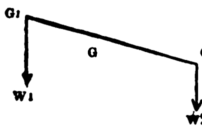
Ex. 193.—Show that the centre of gravity of a cylinder is at the middle point of its axis.

[It may be regarded as evident that the same rule will hold good of any prism].

Ex. 194.—Show that the centre of gravity of a parallelopiped is at the point of intersection of its diagonals.

48. *Centre of gravity of a figure consisting of two or more simple figures.*—Let W_1 and W_2 be the weights of the simple figures and G_1, G_2 their centres of gravity, join G_1G_2, divide it in G in such a manner that

Fig. 39.



$$G_1G : GG_2 :: W_2 : W_1$$

Then is G the required centre of gravity.

If there were a third body weighing W_3 whose centre of gravity is G_3, we can find the common centre of gravity of the three by joining GG_3 and dividing it into parts inversely proportional to W_1+W_2 and W_3; and of course we could continue the same construction to a fourth or a fifth weight, &c.

Ex. 195.—Two spheres whose radii are respectively 4 and 5 in. touch one another; determine the distance of the centre of gravity from the centre of the smaller sphere when the former is of copper and the latter of cast iron. *Ans.* 5·54 in.

Ex. 196.—A cast iron sphere whose radius is 4 in. is fastened to a copper cylinder 3 ft. long, and whose section is 1 in. in diameter; the prolongation of the axis of the cylinder passes through the centre of the sphere. Find the distance between the centre of the sphere and the centre of gravity of the whole. *Ans.* 2·507 in.

Ex. 197.—Determine by construction the centre of gravity of the bodies shown in fig. *c*, where AB is a beam 20 ft. long, and its section 1 ft. square; C and D the centres of two cylinders one foot thick, the radii of whose bases are respectively 6 ft. and 4 ft.; they are of the same material as the beam, and rest with their centres of gravity vertically over the axis of the beam, at distances of 6 in. from A and B respectively.

Construct the figure to scale,—this is done in fig. *c*, to the scale of 1 in. for 5 ft.—join CD, then the weights of the cylinders being in the proportion of 9 to 4, divide CD into parts DG_1 and G_1C respectively proportional to 9 and 4; this will give the centre of gravity of the two cylinders. The construction may be made as follows, by Eucl. bk. VI.—Take DH any line containing 13 equal parts (in the figure each part is $\frac{1}{8}$th of an inch) and measure off DK containing 9 of them, join HC and draw KG_1 parallel to HC;

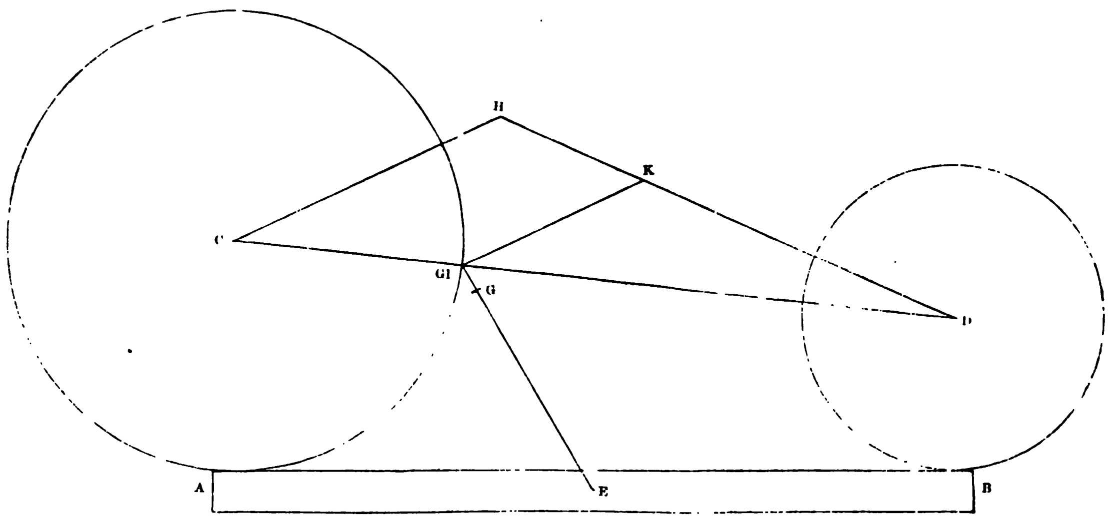


Fig. c. p. 86.

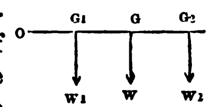
then $CG_1 : G_1D :: HK : KD$ *i. e.* :: 4 : 9. Find E the centre of gravity of the beam, join EG_1; now the united weight of the cylinders is to the weight of the beam very nearly in the ratio 163 : 20, hence, divide EG_1 in G so that $EG : GG_1 :: 163 : 20$, and the point G is the centre of gravity required.

Ex. 198.—A disc of cast iron 12 in. in radius and 2 in. thick rests on a disc of lead 24 in. in radius and 3 in. thick; the circumference of the upper disc passes through the centre of the lower; determine by construction the centre of gravity of the whole.

Ex. 199.—If any quadrilateral be drawn on paper, show that its centre of gravity can be found by construction without the use of a scale.

49. *The centre of gravity of points lying in a straight line.*—The method above explained of finding the centre of gravity of a collection of two or more bodies can be applied to all cases; however if there are only two bodies, or if the centres of gravity of three or more bodies lie in a line, it is commonly more convenient to determine its distance from some fixed point in that line. Let G_1, G_2 be the centres of gravity of the two bodies whose weights are W_1 and W_2 respectively; then the distance GO of the centre of gravity of W_1 and W_2 from O is determined by the equation

Fig. 40.



$$OG(W_1 + W_2) = OG_1 \times W_1 + OG_2 \times W_2.$$

The method of treating three or more weights is exactly the same. It is also plain that if we know OG and OG_2, the same equation will give us OG_1.

Ex. 200.—How far from the one end of the handle is the centre of gravity of the hammer described in Ex. 9 situated, if we suppose the other end to fit square with the face of the hammer?

[If the annexed figure represent the hammer, we have $OA = 42$ in. $AB = 2$ in. so that if G_1 is the centre of gravity of the handle and G that of the head, we have $OG_1 = 21$ in. $OG_2 = 41$ in. Also the weight of the handle is 4·45 lbs. and of the head 9·36 lbs. Hence

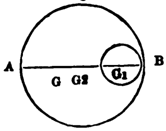
Fig. 41.

$$OG \times 13{\cdot}81 = 21 \times 4{\cdot}45 + 41 \times 9{\cdot}36$$
$$\therefore OG = 34{\cdot}5 \text{ inches}].$$

Ex. 201.—How far from the end of the handle is the position of the centre of gravity of the hammer described in Ex. 12? *Ans.* $72\frac{9}{11}$ in.

Ex. 202.—Let A B be the diameter of a circular disc of cast iron 12 in. in radius; out of the disc is cut a circular hole (whose centre is in A B) 4 in. in radius; the shortest distance between the circumferences is one inch; find the distance of G, the centre of gravity of the remainder, from A. *Ans.* $11\frac{1}{8}$ in.

Fig. 42.

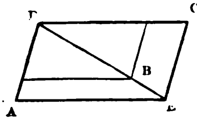


Ex. 203.—If in the last Example the hole were filled up with lead, determine the distance of the centre of gravity of the body from A. *Ans.* 12·42 in.

Ex. 204.—If an oaken cylinder exactly filled the hole in the disc in Ex. (202) and projected 29 in. on each side of the disc, which is 2 in. thick (so that the cylinder is 5 ft. long), find the centre of gravity of the whole. *Ans.* 13·7 in.

Ex. 205.—The gnomon ABC is cut out of a parallelogram A C; determine the distance of its centre of gravity from E; having given that, D E and D B are respectively 20 and 15 ft. in length. *Ans.* 6·786 ft.

Fig. 43.



Ex. 206.—If A B is the axis of a cross made up of six squares each being 3 in. on the side; find the distance of the centre of gravity from A. *Ans.* $6\frac{1}{2}$ in.

Ex. 207.—There are two spheres which are 5 and 6 in. in radii, the larger one of lead and the smaller of cast iron; they are connected by a rod 3 ft. 1 in. long; determine the distance of the centre of gravity of the whole from the centre of the larger sphere,—the weight of the rod being neglected. *Ans.* 12·9 in.

Ex. 208.—In the last Example suppose the rod to weigh 3 lbs. per foot; determine the distance of the centre of gravity of the whole from the centre of the larger sphere. *Ans.* 13·1 in.

Ex. 209.—There is an open cylindrical vessel of lead; it is externally 12 inches high and the base 6 inches in diameter, the thickness of the metal (both of bottom and sides) is $\frac{1}{4}$ of an inch; in it is placed an iron sphere 4 inches in diameter, so that the axis of the cylinder may pass through the centre of the sphere; the vessel is then filled to the brim with water; determine the depth of the centre of gravity below the surface. *Ans.* 6·91 in.

Ex. 210.—A B is a cylindrical rod of steel 40 inches long and $\frac{1}{4}$ of an inch in diameter; at the end, B, is cut a fine screw making 20 turns to the inch; on this is fitted a steel cylinder C D (whose axis coincides with that of the rod) half an inch thick and an inch and a half in radius; determine the distance of the centre of gravity from A when the base of C D is in the same plane as the end B of the rod. *Ans.* 32·66 in.

Ex. 211.—If in the last Example the measurement were made at 60° Fahr., by how much will the centre of gravity fall if the temperature is

raised 30°; and how many turns of CD on the screw will restore the centre of gravity to its former position?

Ans. (1) 0·00625. (2) $\frac{1}{5}$th of a turn.

Ex. 212.—A cone with its vertex downward contains mercury; the depth of the liquid is 6 inches and radius of surface 2 inches; if the temperature rises 50° F., determine the rise in the centre of gravity—neglecting the expansion of the vessel. *Ans.* 0·008 in.

Ex. 213.—A brass rod and a steel rod are fastened at the ends so as to be in one straight line; they are of the same thickness; the length of the brass rod is a; what must be the length of the steel rod, that a change of temperature shall not affect the distance of the centre of gravity from the joint. *Ans.* 1·333a.

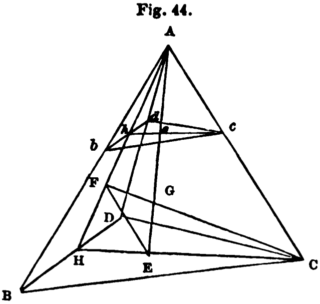
Section II.

50. *Remark.*—The following examples of the determination of the centres of gravity are similar to those contained in the first section, but involve somewhat greater geometrical difficulties; in many cases it will be well if the reader bears in mind, that when bodies are of the same substance, their weights are proportional to their volumes, so that it frequently happens we may reason upon their *volumes* instead of their *weights*.

Ex. 214.—To find the centre of gravity of a triangular pyramid.

Let ABCD be the pyramid; bisect BD in H, join AH and HC; take FH$=\frac{1}{3}$AH and HE$=\frac{1}{3}$HC; draw FC and AE, then these lines being in the same plane, viz. ACH, will intersect, let them do so in G; this point will be the required centre of gravity, and EG will equal $\frac{1}{4}$th part of AE. For draw any plane bcd parallel to BCD cutting the plane ACH in hc, the line AE in e, and AH in h; then h is the middle point of bd; and it is evident by similar triangles that

Fig. 44.



$$he : Ah :: HE : AH$$
$$Ah : hc :: AH : HC$$
$$\therefore \text{(Ex aeq.) } he : hc :: HE : HC$$

but HE$=\frac{1}{3}$HC $\therefore$ $he=\frac{1}{3}hc$, and e is the centre of gravity of the triangle bcd; and the same being true of every other parallel section, the centre of

gravity of the pyramid must be in AE; in precisely the same manner it can be proved that the centre of gravity of the pyramid must be in CF; therefore it must be at G the point of intersection of AE and CF. Next, to show that $EF = \frac{1}{4} AE$. Join FE; then since $HE = \frac{1}{2} EC$ and $HF = \frac{1}{2} FA$, we have $HE : EC :: HF : FA$, and therefore FE is parallel to AC; hence the triangles GEF and GAC are similar, and we have

$$GE : GA :: EF : AC :: EH : CH$$

but $EH = \frac{1}{3} CH$ $\therefore$ $GE = \frac{1}{3} GA = \frac{1}{4} AE$. Hence the centre of gravity of a triangular pyramid is found by the rule:—Join the centre of gravity of the base and the vertex of the pyramid, measure upward from the base a fourth part of this line; the point so found is the centre of gravity of the pyramid.

Ex. 215.—Show that the centre of gravity of any pyramid or cone is found by the same rule as the centre of gravity of a triangular pyramid.

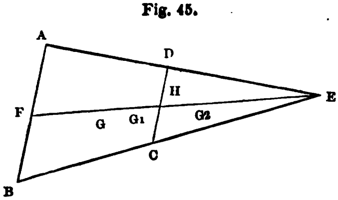
Ex. 216.—If out of any cone a similar cone is cut, so that their axes are in the same line and their bases in the same plane; show that the height of the centre of gravity of the remainder above the base equals $\frac{1}{4} \cdot \frac{h^4 - h'^4}{h^3 - h'^3}$ where h is the height of the original cone, and h' the height of that which is cut away.

Ex. 217.—If out of any right cylinder is cut a cone of the same base and height; show that the centre of gravity of the remainder is $\frac{5}{8}$ of the height above the base.

Ex. 218.—Find the centre of gravity of a trapezoid in terms of the length of the two parallel sides, and of the line joining their middle points.

Let ABCD be the trapezoid, of which AB and CD are the parallel sides; produce AD and BC to meet in E; bisect AB in F, join EF cutting DC in H, which is its middle point. Take $FG_1 = \frac{1}{3} FE$, $HG_2 = \frac{1}{3} HE$; then G_1 is the centre of gravity of the whole triangle ABE, and G_2 of the part CDE; therefore G, the centre of gravity of the remainder, will lie in FE. Now, we have given $AB = a$ $DC = b$ and $FH = h$, and are to find $FG = x$.

Fig. 45.



Since the weights are in the same proportion as the areas of the triangles ABE and CDE, we have

$$FG_1 \times ABE = FG \times ABCD + FG_2 \times CDE$$

$$\text{Now } FG_1 = \tfrac{1}{3} FE \text{ and } FG_2 = h + \tfrac{1}{3} HE = h + \tfrac{1}{3}(FE - h) = \frac{2h}{3} + \tfrac{1}{3} FE.$$

$$\therefore x \times ABCD = \tfrac{1}{3} \cdot FE \times ABE - \left(\frac{2h}{3} + \tfrac{1}{3} FE\right) \times CDE.$$

But by similar triangles (Eucl. 19—VI.).

$$ABE : CDE :: a^2 : b^2$$

$$\therefore ABCD : CDE :: a^2-b^2 : b^2$$

$$\therefore x\,(a^2-b^2)=\tfrac{1}{3}\cdot FE\times a^2-\left(\frac{2h}{3}+\tfrac{1}{3}FE\right)b^2$$

$$=\tfrac{1}{3}\cdot FE\times(a^2-b^2)-\tfrac{2}{3}hb^2$$

Again, by similar triangles

$$FE : HE :: AE : DE :: a : b$$

$$\therefore FE : FE-HE :: a : a-b$$

$$\therefore FE=\frac{ha}{a-b}$$

$$\therefore x\,(a^2-b^2)=\tfrac{1}{3}\,h\,a\,(a+b)-\tfrac{2}{3}\,h\,b^2$$

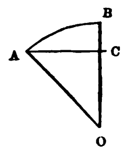
$$=\frac{h}{3}\,(a^2+ab-2\,b^2)$$

$$=\frac{h}{3}\,(a+2b)\,(a-b)$$

$$\therefore x=\frac{h}{3}\cdot\frac{a+2b}{a+b}$$

Ex. 219.—Show that the centre of gravity of the frustum of a pyramid is situated on the line joining the centres of gravity of the ends and at a distance from the lower end, given by the formula $x=\frac{h}{4}\cdot\frac{a^2+2\,ab+3\,b^2}{a^2+ab+b^2}$ where a and b are any pair of homologous sides of the ends, and h is the length of the line joining the centres of gravity of the ends.

Fig. 46.



Ex. 220.—If a segment of a sphere is described by the revolution of A B C round B O; show that the centre of gravity of the surface of the segment is in the middle point of B C.

[It can be easily proved that if B C is divided into any number of equal parts, and planes are drawn perpendicularly through the points of section, they will divide the surface of the segment into equal zones — the weight of each can be collected in B C; and as these weights will be uniformly distributed along B C, the required centre of gravity will be in its middle point].

Ex. 221.—Show that the centre of gravity of the spherical sector formed by the revolution of the sector ABO round BO is at a distance from O= $\frac{3}{4}$OB$-\frac{3}{8}$BC.

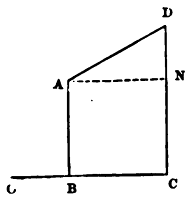
[It must be remembered that the spherical sector may be conceived to be made up of an indefinitely great number of pyramids whose bases form the spherical surface, and having a common vertex O; the weights of each of these can be collected at its centre of gravity, distanced $\frac{3}{4}$ O B from O, and the question is reduced to a case of the last Ex.]

Ex. 222.—Determine the position of the centre of gravity of the volume of the spherical segment formed by the revolution of ABC round BO.

51. *Applications of the formulæ of Prop.* 15.—When a body consists of parts, and we know the weights of the several parts, and the coordinates of their centres of gravity; the coordinates of the centre of gravity of the body will be found by means of the formulæ of Prop. 15.

Ex. 223.—Find the coordinates of the centre of gravity of the trapezoid A B C D, having given O B = 7 ft. O C = 19 ft. A B = 12 ft. D C = 18 ft.; the angles at B and C being right angles.

Fig. 47.

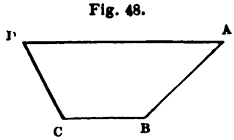


[If A N is drawn parallel to B C dividing the figure into a triangle and a square, the coordinates of the centre of gravity of each can be easily found, and if $\bar{x}$ and $\bar{y}$ are the required coordinates, it will appear that they are determined by the equations

$$180\,\bar{x} = 13 \times 144 + 15 \times 36$$
$$180\,\bar{y} = 6 \times 144 + 14 \times 36.]$$

Ans. $\bar{x} = 13\frac{2}{5}$ $\bar{y} = 7\frac{3}{5}$

Fig. 48.



Ex. 224.—Let A B C D represent the section of a ditch, the breadth A D is 20 ft. and the depth 8ft.; the slope of A B is 1 in 1 and of D C is 2 in 1; determine the horizontal distance from A of the centre of gravity of the section.

Ans. $10\frac{6}{7}$ ft.

Ex. 225.—If in the last Example the breadth A D is a feet, the depth of the ditch h feet, and if A B has a slope of m in 1 and D C of n in 1, show that if $\bar{x}$ be the horizontal distance of the centre of gravity of the section from A; then $\bar{x}$ will be found by the formula

$$\bar{x}\left\{a - \frac{h}{2}\cdot\left(\frac{1}{m} + \frac{1}{n}\right)\right\} = \frac{a^2}{2} - \frac{ah}{2}\frac{1}{n} - \frac{\ }{6}\left(\frac{1}{m^2} - \frac{1}{n^2}\right)$$

Ex. 226.—If A B C D represents the section of a wall of which B C is vertical and equal to h, A B = a and D C = b; then if w is the weight of a cubic foot of the material, the moment of 1 foot of the length of the wall round A and B respectively are given by the formulæ

$$M = \frac{wh\,(2\,a^2 + 2\,ab - b^2)}{6}$$

$$\text{and } M = \frac{wh(a^2 - 3\,ab + 5b^2)}{6}$$

Ex. 227.—The engine-room of a steam vessel is 30 feet long, 20 feet wide, and 15 feet high; at 10 feet from one side, 6 feet from one end, and 5 feet from the floor, is situated the centre of gravity of the boiler, the weight of which

is 2 tons; at 4 feet from the same side, 11 feet from the same end, and 7 feet from the floor, is the centre of gravity of the beam of the engine, which weighs $\frac{1}{2}$ a ton; at 9 feet from the side, 7 feet from the end, and 3 feet from the floor, is the centre of gravity of the furnace, which weighs $1\frac{1}{2}$ ton; at 5 feet from the side, 11 feet from the end, and 10 feet from the floor, is the centre of gravity of the cylinder, which weighs 1 ton: where is the centre of gravity of the whole?

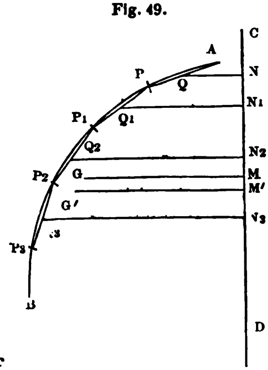
Ans. 8·1 ft. from the side, 7·8 ft. from the end, 5·6 ft. from the floor.

52. *Geometrical applications of the properties of the centre of gravity.*—The most important of these are proved in the following propositions.

Proposition 16.

If a surface be described by the revolution of a plane curve round a fixed axis, its area is found by multiplying the length of the curve into the length of the path described by its centre of gravity.

Fig. 49.



Let AB be the curve, CD the axis of revolution; G the centre of gravity of the curve; draw GM perpendicular to CD; we have to show that the area of the surface described by the revolution AB round CD is found by multiplying the length of AB into the length of the path described by G.

In AB place any number of equal chords, viz. AP, PP_1, P_1P_2, &c. Take $Q, Q_1, Q_2, \ldots$ their middle points, and draw $QN, Q_1N_1, Q_2N_2, \ldots$ perpendicular to CD; also find G′ the centre of gravity of the chords, and draw G′M′ perpendicular to CD; now when the curve revolves round CD, the chords will describe frustums of cones, the surfaces of which will be respectively $2 \times AP \times QN$, $2\pi \times PP_1 \times Q_1N_1$, $2\pi \times P_1P_2 \times N_1N_2$, &c.

and therefore the sum of the surfaces of these frustums will equal

$$2\pi\,(AP \times QN + PP_1 \times Q_1N_1 + P_1P_2 \times Q_2N_2 + \ldots).$$

But by the property of the centre of gravity (Prop. 15) we have

$$G'M'\;(AP + PP_1 + P_1P_2 + \ldots) = AP \times QN + PP_1 \times Q_1N_1 + P_1P_2 \times Q_2N_2 + \ldots$$

Therefore the sum of the surfaces of the conic frustums will equal

$$2\pi\; G'M' \times \text{the sum of the chords } AP,\ PP_1,\ P_1P_2 \ldots.$$

Now this being true, however great the number of chords, will be true of the limits; but the surface of the solid of revolution is the limit of the sum of the surfaces of the conic frustums; the length of the curve is the limit of the sum of the chords; and since G' must ultimately coincide with G, the limit of $G'M'$ is GM. Therefore, volume of surface of revolution $= 2\pi$ GM $\times$ length of curve AB. But 2π GM is the length of the path of G, or the area of the surface is found by multiplying the length of the curve into the length of the path of its centre of gravity.

Cor.—It is manifest that the above proof includes the case of the figure described by the revolution of an area bounded by straight lines. It is also obvious that the same rule applies to any portion of the area contained between two given positions of the revolving curve.

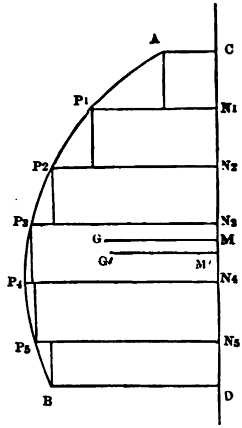
Proposition 17.

If a plane curve revolve about any axis, the volume of the solid described is found by multiplying the area of the curve by the length of the path of its centre of gravity.

Let ABCD be the plane curve; the lines AC and BD

are perpendicular to CD, the axis about which it revolves; find G its centre of gravity, and draw GM perpendicular to CD: we have to show that the volume of the solid described by the revolution of ABCD equals the length of G's path multiplied by the area of ABCD.

Fig. 50.



Divide CD into any number of equal parts in N_1, N_2, N_3, ... and from these points draw ordinates to meet the curve in P_1, P_2, P_3, ... and complete the rectangles AN_1, P_1N_2, P_2N_3, ...; when the figure revolves round CD, these rectangles will describe cylinders, and their united volumes will equal

$$\pi\,(AC^2 \times CN_1 + P_1N_1^2 \times N_1N_2 + P_2N_2^2 \times N_2N_3 + \dots)$$

Let G′ be the centre of gravity of these rectangles, draw G′M′ perpendicular to CD; now the centre of gravity of AN_1 is at a distance from CD equal to $\frac{1}{2}$ AC, that of $P_1\,N_2$ is at a distance from CD equal to $\frac{1}{2}$ $P_1\,N_1$, and similarly of the others. Hence G′M′ × sum of rectangular areas, equals

$$\tfrac{1}{2}AC \times AC \times CN_1 + \tfrac{1}{2}P_1N_1 \times P_1N_1 \times N_1N_2 + \tfrac{1}{2}P_2N_2 \times P_2N_2 \times N_2N_2 + \dots$$

Therefore the sum of the volumes of the cylinders above mentioned will equal

$$2\pi\, G'M' \times \text{the sum of the areas of } AN_1, P_1N_2, P_2N_3, \dots\dots$$

and this being true whatever be the number of parts into which CD is divided, will be true of the limits; now the volume of the solid of revolution is the limit of the sum of the cylinders; the curvilinear area is the limit of the sum

of the rectangles; and since G′ must ultimately coincide with G, the limit of G′M′ is GM. Hence the volume of the solid of revolution is found by multiplying the area of the curve by the length of the path described by its centre of gravity.

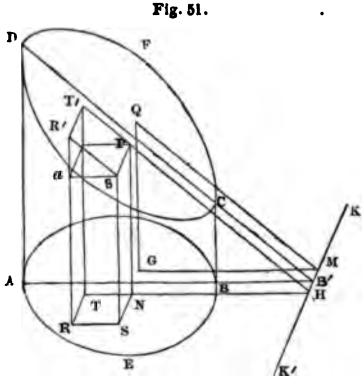
Cor.—The remarks contained in the corollary to the last are applicable, *mutatis mutandis*, to the present Proposition.

Proposition 18.

If a right prism or cylinder is cut by any plane, the volume of the frustum is found by multiplying the area of the base into the length of a line drawn perpendicularly to the base through its centre of gravity, and terminated by the cutting plane.

Let ABCD be the frustum of the right prism or cylinder, standing on the base ABE, whose centre of gravity is G; through G draw GQ at right angles to ABE and terminated by the cutting plane DCF; we have to show that the volume of the frustum is found by multiplying the area of AEB into the length of GQ. Suppose the plane of the paper to be perpendicular to the planes of the ends, and to cut them in ABB′CD; if the planes of the two ends are produced, they will intersect in a line KK′ perpendicular to the plane of the paper; hence AB′D is the angle of inclination of the cutting plane to the base; we will denote this angle by θ. Draw GM at right angles to KK′.

Fig. 51.



In AEB describe a series of rectangles of equal width, with sides perpendicular to AB; and divide them into squares, each of whose areas is represented by A; and of these squares, let NSRT be one; from the points NRST draw lines perpendicular to the base, and terminated by the cutting plane, viz. PN, RR′, SS′, TT′; through PS′ draw a plane Pa parallel to the base, the figure PTaS is a rectangular parallelopiped, and its volume is equal to $A \times PN$. Now all the angles at N being right angles, the plane PT is perpendicular to SN, and therefore if produced to cut KK′, it will be perpendicular to KK′, let it cut that line in H; then T′P and TN when produced, will meet in H and each will be perpendicular to KK′; therefore, the angle PHN equals θ, and the volume of PTaS equal

$$A \times HN. \tan \theta$$

Now if we imagine the same to be done on each of the squares, we shall have the required volume equal to $\tan \theta \times$ the limit of the sum of each square area multiplied by the perpendicular distance of an angle from KK′. But in the limit, the magnitude of the side of each square can be neglected in comparison with its distance from HH′; hence the above limit will be the same as the limit of the sum of the areas of the squares, into the distance of the centre of gravity from KK′, *i.e.* will equal area $AEB \times GM$, and the volume will equal $AEB \times GM \tan \theta$.

Now if QM be joined $QMG = \theta$, therefore $GQ = QM \tan \theta$, and the required volume equals area $AEB \times GQ$.

Cor.—It is evident that if the prism or cylinder is cut by another plane inclined at any angle to the base, the volume contained between them equals the area of the perpendicular section multiplied into the part contained between the planes of a line drawn through the centre of gravity of that section at right angles to its plane.

Ex. 228.—Show that propositions 16 and 17 are true in the case when the curve is a closed curve and revolves round an axis wholly without it.

Ex. 229.—In Proposition 18 show that Q is the centre of gravity of DCF.

Ex. 230.—An equilateral triangle revolves round its base, whose length is a; find the area and volume of the figure described.

Ans. (1) $\pi a^2 \sqrt{3}$ (2) $\frac{\pi a^3}{4}$

Ex. 231.—An equilateral triangle revolves round an axis parallel to the base, the vertex of the triangle being between the axis and the base, the base is 6 in. long and the distance from the vertex to the axis is 9 in.; determine the volume of the ring described. *Ans.* 1220·7 cub. in.

Ex. 232.—Determine the volume of a ring formed like that in the last example having given that each side of the triangle is 6 in. and the external diameter of the ring 3 ft. *Ans.* 1593·4 cub. in.

Ex. 233.—The section of a ring is a trapezoid, its height is 3 in. and its parallel sides are respectively 7 in. and 3 in. long, they are parallel to the axis, the shorter being the nearer to the axis and at a distance of 11 in.; find the volume of the ring. *Ans.* 1196·9 cub. in.

Ex. 234.—In the last Example if the longer side of the trapezoid had been the nearer to the axis, the external diameter of the ring being the same in both cases, what would have been the volume? *Ans.* 1159·2 cub. in.

Ex. 235.—Determine the volume and surface of a ring with a circular section whose internal diameter is 12 in. and thickness 8 in.

Ans. (1) 333·1 cub. in. (2) 444·1 sq. in.

Ex. 236.—Determine the volume and surface of a ring whose section is a regular hexagon, whose circumscribing circle has a radius a and its centre at a distance b from the axis of revolution.

Ans. (1) $3\pi b a^2 \sqrt{3}$. (2) $12 \pi a b$.

Ex. 237.—Find the centre of gravity of the arc of a semicircle.

Ans. Distance from centre $= \frac{\text{diam.}}{\pi}$

Ex. 238.—Find the centre of gravity of the area of a semicircle.

Ans. Distance from centre $= \frac{2}{3}.\frac{\text{diam.}}{\pi}$

Ex. 239.—A cylindrical shaft is cut off obliquely at an angle of 45° to the axis, its radius is 6 in. and its extreme height is 2 ft. 6 in. Find its solid contents. *Ans.* 1·5708 cub. ft.

Ex. 240.—A cylindrical shaft is cut obliquely at an angle of 30° to the axis, the radius of the base is 10 in., the extreme height of the shaft 3 ft.; find its volume. *Ans.* 9497 cub. in.

Ex. 241.—A right prism stands on a triangular base the angles of which are ABC, the angles of the other end being DEF, the sides AB, AC are each 15 ft. long, BC is 18 ft. long; the other edges, viz. AD, BE, CF are each

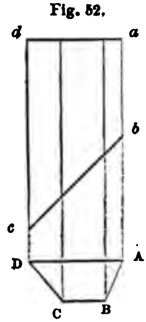
30 ft. long; through the edge BC passes a plane making an angle of 60° with the base; determine the volumes of the parts into which the prism is divided. Also if the prism were cut by a plane parallel to the former and cutting AD at a distance of 24 ft. above A, find the volumes of the two parts.

Ans. (1) 748·3 and 2491·7 cub. ft. (2) 1095·6 and 2144·4 cub. ft.

Ex. 242.—Show that if any triangular prism be cut by a plane so that the edges perpendicular to the base are respectively a, b, c, and the area of the base A, then the volume of the frustum will be $\frac{1}{3}$ A $(a+b+c)$.

Ex. 243.—Let $abcd$ represent the plan and ABCD the section of a portion of a ditch; AD = 20 ft.; depth of ditch 8 ft.; slope of AB is 2 in 1, and that of DC is 1 in 1; ab and cd are respectively 20 and 40 ft. long. Find the volume; and determine the error that would be committed if we had found the volume by multiplying the area of the section by half the sum of ab and dc.

Fig. 52.



Ans. (1) 3264 cub. ft. (2) Error 96 cub. ft.

[Compare Ex. 224.]

Ex. 244.—Let ABCD be the plan of square redoubt each side of which is 150 ft., the corners of the ditch are quadrants of circles whose centres are respectively A, B, C, D, So that it has a uniform width of 24 ft., its depth is 9 ft., the inside slope is 3 in 1 and the outside 1 in 1. Find its volume. *Ans.* 108057 cub. ft.

Ex. 245.—If the ditch in the last Example were surrounded with a glacis 3 ft. high whose outside slope is 1 in 10 and inside slope 1 in 1; find its volume, *Ans.* 40897 cub. ft.

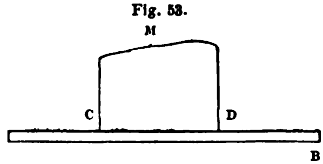
CHAP. V.

FRICTION OF PLANE SURFACES—INCLINED PLANE, WEDGE, SCREW.

SECTION I.

53. *Reaction of surfaces.* — It nearly always happens that amongst the pressures which keep a body at rest is the reaction of one or more surfaces; to explain the nature of this reaction let us consider a particular case: suppose a mass M to rest on a table AB, and suppose it to weigh 1000 lbs.; that weight must be supported by the table, which must therefore exert upwards a pressure of 1000 lbs. in a direction equal and opposite to the direction of the weight.

Fig. 53.

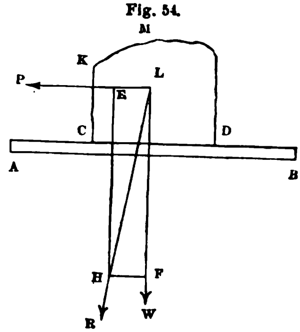


It we consider the case particularly we shall see that this reaction is an instance of a *distributed* pressure, for the under surface of CD will be in contact with the table at many points, and at each point there will be a reaction; what is the magnitude of the reactions at the different points we do not commonly know, they must however be such that their resultant shall act vertically upward through the centre of gravity of M and shall equal 1000 lbs. And, in general, if a body is at rest when pressed against a surface the various points of that surface must supply reac-

tions whose resultant is equal and opposite to the resultant of the pressures by which that body is urged against the surface; this resultant reaction is called *the reaction* of the *surface.*

54. *The limiting angle of resistance.*—The question now arises under what circumstances is the plane capable of supplying the reaction necessary to produce equilibrium? this will be the case if the plane does not break, and if it keeps the body from sliding; it is with the latter condition we are here concerned. Let us revert to the example discussed in the last article, and let us suppose a rope to be fastened to the point K by means of which the body is pulled horizontally by a pressure P; we know that if P have a certain magnitude it will just make the body slide, but if it be less than that certain magnitude the body will continue at rest; suppose that a pressure of 190lbs. will just not make the body slide; produce PK to meet the vertical through the centre of gravity in L, let LE represent P (190) and LF represent W (1000), complete the parallelogram LH, this must be the direction of the resultant pressure R, and its direction makes with a perpendicular to AB an angle of 10° 45′; now if the pressure P is less than 190lbs. the resultant pressure will fall within the angle RLW; but if it be greater than 190lbs. it will fall without the angle RLW; in the former case the surface AB can supply a reaction which prevents motion, in the latter it cannot; and thus in the case we have supposed the surface AB can supply a reaction in any required direction which makes an angle less than 10° 45′ with the normal, *i. e.* the perpendicular, to the surface; and when the body is in the state

Fig. 54.



bordering on motion, the direction of the reaction will make an angle equal to 10°45′ with the normal.

Now it appears from experiment that if the surface AB were of cast iron, and the mass M of wrought iron, a pressure of 190 lbs. would be required just not to produce motion in the case above discussed; and it also appears from experiment that within very considerable limits, the same proportions are preserved, irrespective of the *extent* of the surface pressed and the amount of the pressure; so that we may state as a fact of experience that when wrought iron rests on cast iron the latter will exert a reaction in any direction required to produce equilibrium that does not make with the normal an angle greater than 10° 45′, and when motion is about to ensue, the direction of the reaction will make an angle with the normal, equal to 10° 45′, this angle is therefore called the *limiting angle of resistance* in the case of cast iron upon wrought. It further appears from experiment, that in the case of any two surfaces whatever, there is a limiting angle of resistance proper to those surfaces, and depending on their physical character; for instance, in the case of wrought iron on oak, the angle is 31° 50′, and similarly in other cases. Values of this angle in several cases are given in Table XI.

Hence, when one body is pressed upon another by certain forces, the latter will react upon it in a direction opposite to the resultant of those pressures, provided it makes with the perpendicular to the surface of contact an angle less than the limiting angle of resistance; and when the pressures are such as to bring the body into the state bordering on motion, the direction of the reaction will be inclined to the perpendicular at an angle equal to the limiting angle of resistance. It is evident that the reaction will always act so as to oppose the motion of the body.

Ex. 246.—A mass of oak rests endwise on a horizontal oak floor; it weighs 750 lbs. and is pulled by means of a rope whose direction is inclined

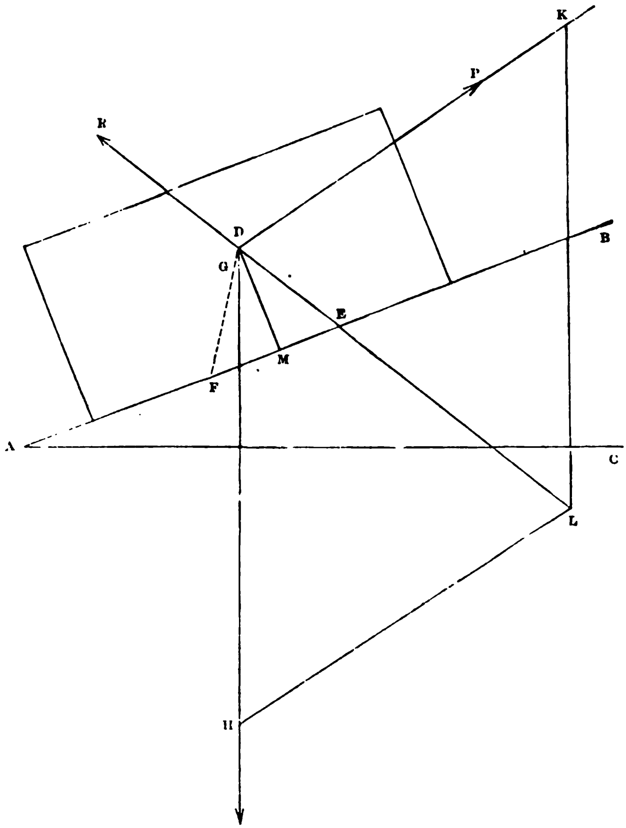
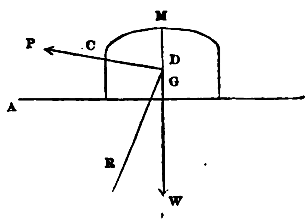


Fig. *d.* p. 103.

to the horizon at an angle of 10°; find the pressure that will be on the point of making the body slide, and also the total pressure on the floor.

[Let AB be the floor, M the mass, CP the direction of the rope, through G the centre of gravity of the mass draw the vertical line M W, produce PC to meet MW in D, and draw the line DR making with MW an angle RDW equal to 23° 20′, this will be the direction of the reaction in the case we have supposed, and now P and R can be found by the parallelogram of pressures.]

Fig. 55.



Ans. P = 305·3 lbs. R = 759 lbs.

Ex. 247.—In the last Example determine P and R if the mass M is of wrought iron, and the direction of the rope inclined to the horizon at an angle of 15°. *Ans.* P = 413·3 lbs. R = 756·9 lbs.

Ex. 248.—What would be the required pressure P. in the last case if the direction of the rope were horizontal. *Ans.* P = 465 lbs.

Ex. 249.—Determine the angle made by P's direction with the horizon when P is the *least* pressure that will just not make the body slide.

[The required direction must be perpendicular to DR.]

Ex. 250.—Show that when a body rests on a horizontal plane the smallest pressure that will bring it into the state bordering on motion will act in a direction inclined to the horizon at an angle equal to the limiting angle of resistance.

Ex. 251.—A mass of wrought iron weighing 500 lbs. rests on a plane of oak inclined at an angle of 20° to the horizon, a pressure P acts upon it so as just not to pull it up the plane in a direction inclined to the plane at an angle of 12°; find P. (See fig. *d.*)

Let AB be the plane, G the centre of gravity of the mass; through G draw the vertical line DW; produce P's direction to cut this line in D; draw DM perpendicular to AB; make the angle MDE equal to 31° 50′ (the limiting angle of resistance), this line is that along which the reaction acts; and on completing the construction in the usual manner by the parallelogram of pressures the value of P will be found. In fig. (*d*) one inch represents 200 lbs.; so that in the figure from which it was copied DH was equal to $2\frac{1}{2}$ in. And on completing the parallelogram HK it was found that P equals 415 lbs. Calculation gives P equal to 417·9 lbs. It may be remarked that by reason of the uncertainty of the limiting angle of resistance it cannot be affirmed that one of these results is more accurate than the other.

Ex. 252.—In the last Example suppose P to act along PD as a pushing force; find its magnitude that it may just not push the body down the plane.

[If, in fig. *d*, a line DF be drawn making the angle FDM equal to 31°50′; this will now be the direction of the reaction.] *Ans.* 142·1 lbs.

Ex. 253.—Referring to Examples 251 and 252: first, if P had been a pressure of 200 lbs. acting up the plane; next if P had been a pressure of 100 lbs. acting down the plane; and lastly if there were no pressure P; find the magnitude and direction of the reaction of the plane.

Ans. (1) 429 lbs. PDR = 98° 42′. (2) 559·4 lbs. PDR = 49° 17′.
(3) 500 lbs. acting vertically upward.

Ex. 254.—A mass of elm weighing 340 lbs. rests on a plane of oak inclined to the horizon at an angle of 15°; the fibres of the woods are parallel; determine the pressure which will just not pull the body down, first when its direction is inclined upward from the horizon at an angle of 10°; next when its line of action is parallel to the plane; and lastly its magnitude and direction when the least possible. *Ans.* (1) 116·1 lbs. (2) 139·1 lbs.
(3) 114·4 lbs. at an angle of 19°·40′ above the hor.

[For determining when the pressure is the least possible refer to Example 249 and the note upon it.]

Ex. 255.—In the last case determine the magnitude and direction of the least pressure that will drag the body up the plane; and show that in any case the least pressure must be inclined to the horizon at an angle equal to the inclination of the plane *plus* the limiting angle of resistance.

Ans. 259 lbs.

Ex. 256.—What is the least pressure that will draw a cubic foot of cast iron down a plane of oak a inclined to the horizon at an angle of 14°.

Ans. 146·7 lbs.

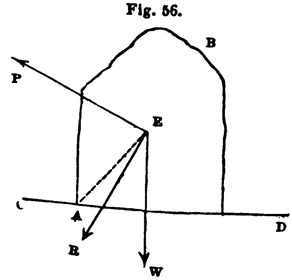
Ex. 257.—In the last Example what would have been the least pressure necessary to support the mass had the plane been of cast iron.

Ans. 38·6 lbs.

Ex. 258.—What would be the horizontal pressure that would just push the body *up* the inclined plane in the last case? *Ans.* 192 lbs.

55. *Conditions under which a body acted on by certain pressures will neither be overthrown nor slide.*— Let a mass AB rest on a horizontal plane CD, and let the pressures concerned be its weight acting vertically along the line EW and a pressure P acting along the line PE: find R the resultant of these pressures; in order that the body may be at rest it is necessary that R be balanced by a reaction equal and opposite to it; this cannot happen if the direction of R cut CD outside the

Fig. 56.



base; hence the condition that the body be not overthrown is that the direction of the resultant pressure fall within the base; if this condition be fulfilled, the body will slide or not, according as the direction of R makes with the normal to the point where it cuts the surface, an angle greater or less than the limiting angle of resistance. The question may be asked, if AB be pulled along the line PE by a continually increasing pressure, will it slide before it topples, or *vice versâ?* This is readily answered by joining AE; then if AEW is less than the limiting angle of resistance, the body will topple before it slides, since R's direction will fall without the base before its direction makes with the perpendicular an angle greater than the limiting angle of resistance; if however AEW is greater than the limiting angle of resistance, the body will slide before it topples. In the intermediate case, when AEW equals the limiting angle of resistance, the body will be on the point of toppling and sliding for the same value of P.

Ex. 259.—A rectangular mass of oak the base of which is 2 ft. square and height 7 ft. rests endwise on a floor of oak, a rope is fastened to it at a certain height above the floor and is pulled by a certain pressure in a direction inclined at an angle of 20° to the horizon; it is found to be on the point both of toppling and sliding; find the height of the point of attachment from the floor and the magnitude of the pressure.

Ans. (1) 2·68 ft. (2) 648·7 lbs.

[It is manifest, referring to the figure in Art. 55, that E will be found by making the angle EAD equal to the complement of the limiting angle of resistance, when the circumstances are those mentioned in the question.]

Ex. 260.—A cylinder of copper the radius of whose base is 2 in. and height $3\frac{1}{2}$ in. rests on a horizontal oak table it is pulled by a horizontal pressure whose direction coincides with a radius of the upper end; find the pressure that will just make the body move, and determine whether the motion will be one of sliding or of toppling.

Ans. (1) The body will topple. (2) 8 lbs.

Ex. 261.—Work the last Example supposing the cylinder to be of oak the fibres being parallel to the axis of the cylinder.

Ans. (1) The body will slide. (2) 10·2 oz.

Ex. 262.—A rectangular mass of cast iron rests on an inclined plane of oak; it is on the point both of sliding down, and also of overturning, its base is 2 ft. square; what is its height? *Ans.* 3·08 ft.

Ex. 263.—In the last Example what pressure acting parallel to the inclined plane would be just sufficient to draw the mass of iron up it? could this pressure be applied at any point of the body so far above the plane as to overturn the body before making it slide up the plane?

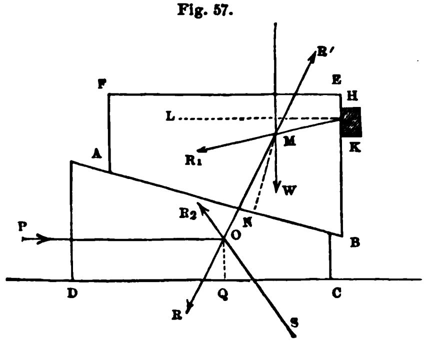
Ans. (1) 6044 lbs. (2) It will overturn the body if applied at a point more than 1·54 ft. above the plane.

56. *Mutual pressure of two moveable bodies.*—In all the above examples we have considered the case of a moveable body resting on a fixed plane; if we suppose both bodies to be moveable we do not introduce any new condition; each will press on the other, and these mutual pressures must be equal and opposite; if the bodies are on the point of sliding upon the surface of contact, the mutual pressures will act so as to oppose the motion, and their direction will be inclined to the normal to the surface of contact, at an angle equal to the limiting angle of resistance.

Ex. 264.—There are two masses of cast iron ABCD, ABEF (as shown in the figure), having a common surface AB inclined at an angle of 15° to the horizon, the lower rests on an oaken table CD (fibres parallel) and is urged forward by a horizontal pressure P of 700 lbs. the upper block ABEF can only move upward in consequence of an obstacle of wrought iron whose section is HK; determine W when on the point of being forced up by P, the pressures against CD and HK, and the mutual pressure on the surface AB—the weight of the masses being neglected.

[Through *any* point of HK draw a line HL perpendicular to EB, then since ABEF is on the point of sliding up the reaction R_1 of HK will act as shown in the figure, the angle LHR_1 being equal to 10° 45′; produce W's direction to cut HR_1 in M, then the direction of the reaction R′ of the surface AB must pass through M; draw MN perpendicular to AB and make the angle NMO equal to 9° 5′, then since sliding is about to ensue on AB this must be the direction of R′ and therefore of R, the reaction of the upper surface

Fig. 57.



on the lower; let P's direction cut R's in the point O and draw OQ perpendicular to CD, then since ABCD is about to slide forward if we make the angle QOS equal to 33° this will be the direction of R_2 the reaction of CD. Thus we have obtained the directions of all the pressures, and knowing P we can determine R and R_2 by the parallelogram of pressures; also since R′ equals R, we can determine W and R_1 in the same manner. The student must observe that since ABCD is about to move *forward* on the surface AB the direction of R must be drawn as in the figure; he must observe that the relation between the pressures does not depend upon the particular point of HK through which we suppose R_1 to act; he will also find it an instructive exercise to consider the change in the construction that would be introduced by taking into account the weights of the masses.]

Ans. W = 584·3 lbs. R = 699·3. R_1 = 290·5. R_2 = 761·3.

Ex. 265.—In the last Example find the value of W which will be on the point of overcoming P, the inclination of AB being 45°.

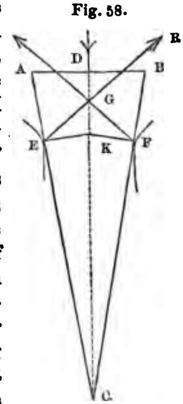
Ans. 10628 lbs.

[The solution is precisely like that of the last example except that since all the surfaces are on the point of sliding in the opposite directions the directions of the reactions will fall on the other sides of the normals.]

Ex. 266.—Show that no value of W however great can force P out, unless the inclination of AB exceeds the sum of the limiting angles of resistance on the surfaces AB and CD.

Ex. 267.—An isosceles wedge of wrought iron with an angle of 10° is forced between two masses of oak; a pressure of 200 lbs. being applied on the back of the wedge is on the point of moving it forward; determine the pressure on the sides.

Fig. 58.



[Let ABC be the wedge; the dimensions are indifferent provided the angle C have the specified magnitude, viz. 10°; draw CD perpendicular to AB, the driving pressure acts along this line; let E be the point of contract of AC and the mass draw EK perpendicular to AC, and make the angle KEG equal to the limiting angle of resistance, viz. 31° 50′ this will be the direction of R the reaction of the mass when the wedge is on the point of sliding forward; then if KF be drawn perpendicular to BC, F must be the point of contract of BC with the other mass of oak, since the reaction of that mass must pass through G and must make an angle of 31° 15′ with the perpendicular, which will be true of FG and of no other line; we thus have three pressures in equilibrium acting through G in known directions, and can determine R as usual—R is (of course) equal and opposite to the pressure on the masses.]

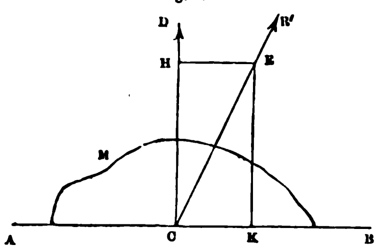
Ans. R = 166·8 lbs.

Ex. 268.—A wrought iron wedge whose angle 13° is driven into a mass of oak by a pressure of 1 cwt., what pressure will it produce on the sides, of the mass? Determine also the pressure if the angle were of 18°.

Ans **(1) 90·3 lbs. (2) 85·6 lbs.**

57. *Friction.*—Let AB be a horizontal table; M a mass which, in consequence of the action of certain pressures, is on the point of sliding in the direction BA; then the reaction R′ will be equal to their resultant, and its direction will be inclined to the perpendicular to AB at an angle ϕ equal to the limiting angle of resistance; let CR′ be the direction of this reaction; draw CD perpendicular to AB, then the angle DCR′ is equal to ϕ; take CE to represent R′, and complete the rectangle HK; we may replace R′ by two components R and F, of which R acts along CD and F along AB; these components are represented by CH and CK respectively; now it is evident

Fig. 59.



that $\tan\phi = \frac{HE}{CH}$ · *i.e.* $\tan\phi = \frac{F}{R}$

$$\therefore F = R \tan\phi$$

The tangential reaction F is commonly called the *Friction*, and $\tan\phi$ (which is generally denoted by the letter μ) is called the *coefficient of friction;* so that when a body resting on a plane is in the state bordering on motion the friction equals the normal pressure multiplied by the coefficient of friction; it will be remarked that unless the body is in the state bordering on motion the whole of the friction is not called into play, but only so much of it as is sufficient to produce equilibrium. In order to complete our remarks on this subject, it is to be observed that when the body actually slides its motion is opposed by a constant

friction which is properly represented by μ times the normal pressure; it appears, however, that the numerical value of μ for the same substance is different in the cases of motion and of rest. The difference is most conspicuous in the case of soft substances (*e.g.* various kinds of wood) that have been some time in contact; wherever a difference exists, the value of μ for substances at rest is larger than the value for the same substances in motion.

58. *Experiments on friction.*—The chief general results that have been elicited by experiments on the friction of surfaces, are called the *laws of friction*, and may be thus stated:

(1.) Friction is proportional to the normal pressure.
(2.) It is independent of the extent of the surfaces in contact.
(3.) In the case of motion, it is independent of the velocity.
(4.) If unguents are interposed between the surfaces of contact, the friction depends mainly on the nature and quantity of the unguent.

It must be added that these laws depend entirely on experimental evidence, and that the first of them ceases to be true when the pressure per square inch becomes very great. The accurate determination of the values of μ, the coefficient of friction for different substances, is due to General Morin, on whose authority the results rest that are registered in Table XI. p. 111.*

* The establishment of the laws of friction appears to be due to Coulomb, whose Memoir of Friction was published in 1785; a very full abstract of the paper is given in Dr. Young's Natural Philosophy, vol. ii. p. 170 (1st ed.). The properties of the limiting angle of resistance and its importance in the statement of mechanical formulæ were first pointed out by Mr. Moseley. General Morin's Tables are very extensive, they have been printed several times; a sufficient account of the experiments on which they are based, together with the Tables themselves, will be found

59. *Smooth surfaces.*— If a surface were perfectly smooth it would be incapable of supplying any tangential resistance to the motion of a body upon it, in this case, therefore, $\mu = o$ and $\phi = o$, or the reaction would be limited to one direction, viz. that of the perpendicular to the surface of contact; it will be a useful exercise for the student to determine the answers to the examples contained in the previous part of this chapter upon the supposition that the surfaces of contact are perfectly smooth; he will find that the answers to the respective questions are as follows:

(*a*) Ex. 246. *Ans.* Any pressure however small.
(*b*) Ex. 248. *Ans.* do. do.
(*c*) Ex. 251. *Ans.* 174·8 lbs.
(*d*) Ex. 255. *Ans.* (1) 88 lbs. (2) Parallel to the plane.
(*e*) Ex. 257. *Ans.* 109 lbs.
(*f*) Ex. 258. *Ans.* 109 lbs.
(*g*) Ex. 264. *Ans.* 2612·5 lbs.
(*h*) Ex. 267. *Ans.* 1147·5 lbs.

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in his work, Notions Fondamentales de Mécanique. To enable the reader to form some conception of the limits within which the laws of friction hold good, the following (somewhat favourable) instance may be adduced. The coefficient of friction is given in the tables as 0·54 in the case of oak resting in the state bordering on motion on oak with the fibres perpendicular to each other, the experimental results from which this value was deduced are as follows :—

Surface of contact.	Normal pressure.	Pressure on point of causing motion.	Coef. friction μ.
0·947 ft.	121 lbs.	67 lbs.	0·55
	283 „	151 „	0·53
	495 „	252 „	0·51
	1995 „	1171 „	0·58
	2525 „	1287 „	0·51
0·043 ft.	389 „	204 „	0·52
	403 „	213 „	0·53
	1461 „	855 „	0·52

TABLE XI.

Coefficients of Friction

And the limiting angles of resistance of substances between which no unguents are interposed.

Substance.	Disposition of fibres.	State bordering on motion.			State of motion.		
		ϕ	μ or tan ϕ.	Sin ϕ.	ϕ	μ or tan ϕ	Sin ϕ.
Oak on Oak . .	Parallel	31°50′	0·62	0·53	25°40′	0·48	0·43
"	Perpendicular	28°20′	0·54	0·47	18°45′	0·34	0·32
"	Endwise	23°20′	0·43	0·40	10°45′	0·19	0·19
Oak on Elm . .	Parallel	20°50′	0·38	0·35			
Elm on Oak . .	Parallel	34°40′	0·69	0·57	23°20′	0·43	0·40
"	Perpendicular	29°40′	0·57	0·50	24°15′	0·45	0·41
Wrought Iron on Oak	Parallel	31°50′	0·62	0·53	31°50′	0·62	0·53
Cast Iron on Oak	Parallel	33°0′	0·65	0·55			
Copper on Oak .	Parallel	31°50′	0·62	0·53	31°50′	0·62	0·53
Wrought Iron on Cast	—	10°45′	0·19	0·19	10°10′	0·18	0·18
Cast Iron on cast .	—	9°5	0·16	0·16	8°30′	0·15	0·15
Oak on calcareous oolite * . . .	Endwise	32°10′	0·63	0·53	2°50′	0·38	0·35
Wrought Iron, do.	—	26°10′	0·49	0·44	3°40′	0·69	0·57
Brick, do. . . .	—	33°50′	0·67	0·56			
Calcareous oolite on do. . . .	—	36°30′	0·47	0·59	32°40′	0·64	0·54

* The stone employed in M. Morin's experiments seems to have been a soft oolitic stone from the quarries at Jaumont near Metz; the nearest English equivalent is probably *Portland stone.*

It is to be observed that in the above Table the numerical values of μ were ascertained by experiment; the values of ϕ and sin ϕ have been obtained by calculation. General Morin's Tables give the values of μ corresponding to various unguents: of these, the following comprehensive result will be sufficient for our present purposes:—any two of the following substances, oak, elm, cast iron, wrought iron, bronze, pressed against each other, tallow being employed as an unguent, have for the coefficient of friction $\mu = 0{\cdot}10$, and therefore $\phi = 5° \; 40'$ and sin $\phi = 0{\cdot}10$. The same substances when in motion, and the unguent is either tallow, hog's lard, soft gom, or any similar substance, have the coefficient of friction equal to 0·07, and therefore $\phi = 4°$ and sin $\phi = 0{\cdot}07$.

Section II.

60. *The Inclined Plane and Wedge.*—The examples which follow on the inclined plane and wedge are precisely similar to those already given in the first section of the present chapter.

Ex. 269.—A mass whose weight is W rests on a plane inclined at an angle α to the horizon; it is supported by a pressure P, the direction of which makes an angle β with the inclined plane; if P is on the point of making W move up the plane show that $P : W :: \sin(\alpha+\phi) : \cos(\beta-\phi)$, and that $R : W :: \cos(\alpha+\beta) : \cos(\beta-\phi)$; where R is the reaction and ϕ the limiting angle of resistance.

[Referring to fig. *d*, it is evident that $RDW=180-(\alpha+\phi)$ and $WDP=90+\phi-\beta$].

Ex. 270.—If in the last example P be such that the mass W is on the point of sliding down the plane, show that

$$P : W :: \sin(\alpha-\phi) : \cos(\beta+\alpha)$$
$$R : W :: \cos(\alpha+\beta) : \cos(\beta+\phi).$$

Ex. 271.—If the body represented in fig. *d* is a cylinder the radius of whose base is *r* and height $2h$, and if P acts at a point X so chosen that for the same value of P the body is on the point of turning round K when it is also on the point of sliding up the plane, show that

$$XK=\frac{(r\cos.\ \alpha+h\sin\alpha)\cos(\beta-\phi)}{\cos\beta\sin(\alpha+\phi)}$$

and transform the expression into one adapted for logarithmic calculation.

Ex. 272.—The earliest experiments on friction were made in the following manner; The substances were formed into rectangular blocks—shaped like bricks—and were placed on planes of various substances; the planes were then gradually raised, and the angles noted at which sliding commenced; it was found that for the same substances this angle was the same whatever the weight of the block, and whether it rested on its broad or narrow face; what conclusions could be inferred from these facts as to the nature of friction?

Ex. 273.—Given an incline of 1 in *n, and that a body weighing W lbs.**

---

* *i. e.* $\frac{1}{n}$ equals the tangent of the inclination.

rests upon it if the friction is 1 lb. in m, show that the pressure which will bring the body into the state bordering on motion up the plane equals $\left(W \frac{1}{n} + \frac{1}{m}\right)$ very nearly. If the incline is 1 in 50 and the friction 1 in 100, show that the error does not exceed the $\frac{1}{100000}$th part of the weight.

Ex. 274.—An isosceles wedge, of which the angle is 2α, is urged by a pressure P between two masses and is on the point of moving forward; if ϕ is the limiting angle of resistance between the sides of the wedge and the masses, show that the pressure exerted by P against the side of each mass is

$$R = \frac{W}{2 \sin(\alpha + \phi)}$$

[See Ex. 267.]

Ex. 275.—Show that if W is the pressure that has forced a wedge into a given position and W_1 the pressure required to extract it*, then

$$W_1 = W \frac{\sin(\phi - \alpha)}{\sin(\phi + \alpha)}$$

Ex. 276.—An iron wedge whose vertical angle is 13° is driven into a mass of oak by a pressure of 1 cwt:—what force will be necessary to extract it? *Ans.* 77·27 lbs

* The result assigned in Ex. 275 presupposes that the tendency of the substance to collapse acts in a direction perpendicular to the sides of the

Fig. 60.

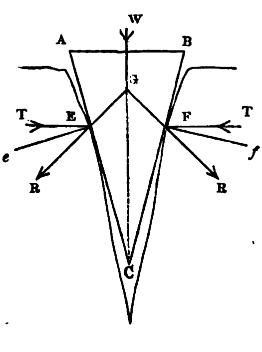
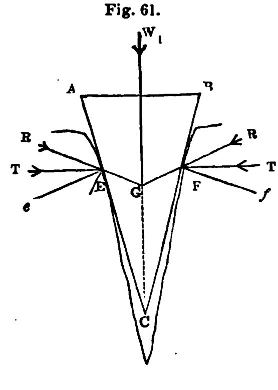


Fig. 61.



wedge; if this is not the case the solution is as follows. Draw eE and fF perpendicular to AC and BC; and let T, the tendency to collapse, act along

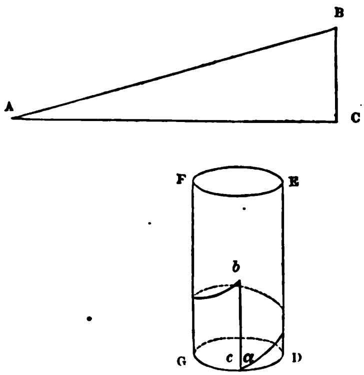
Ex. 277.—Show that the wedge will start if the pressure be withdrawn provided the angle of the wedge be greater than $2\ \phi$.

Ex. 278.—An iron wedge whose angle is $7°$ is driven into a mass of oak, find what fraction of the driving pressure is consumed by friction.

Ans. If W' is the pressure on the smooth wedge which exercises the same normal pressure on the block as is produced by W on the rough wedge, then $W' = 0{\cdot}106\ W$.

61. *The form of the helix or the thread of the screw.*—Let ABC be a right-angled triangle, and DEFG a cylinder, the circumference of whose base is equal to the base of the triangle AC; if we suppose this triangle to be wrapped round the cylinder so that A and C come together, as indicated by the small letters *ac b*, the hypothenuse AB will take the form of a curve called the helix, *i. e.* the curve to which the thread of a screw would be reduced if it became a single line.

Fig. 62.



TE and let $e\,ET = \iota$, then, in figure 60, if W is on the point of making the wedge move forward we have

$$W = 2R\sin(\phi+\alpha)$$
$$R\cos(\phi+\ ) = T^{1}$$

Now if W_1 is the pressure required to extract the wedge from the position into which W has forced it, it is evident that T (which depends upon the shape, &c. of the cleft) is unchanged, the pressures therefore act as shown in figure 61, and we obtain the equations:

$$W_1 = 2\ R_T \sin(\phi-\alpha)$$
$$R\cos(\phi-\iota) = T$$
$$\therefore\ W_1 = W\,\frac{\sin(\phi-\alpha)\cos(\phi+\iota)}{\sin(\phi+\alpha)\cos(\phi-\iota)}$$

The angle ι is commonly unknown, and very small, and therefore is generally neglected

Ex. 279.—If the distance between two turns of a thread of a screw is h and the radius of the cylinder is r, show that the length of n turns of the thread is $n\sqrt{4\pi^2r^2+h^2}$.

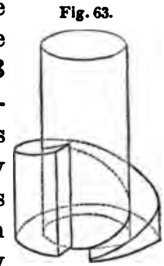
Ex. 280.—Show that if h is the distance between two turns of the thread, and r the radius of the cylinder, then if θ is the angle of inclination of the thread of the screw we shall have $\tan\theta = \dfrac{h}{2\pi r}$

Ex. 281.—The length of a screw is $1\frac{1}{2}$ ft. in which space the screw makes 36 turns, the radius of the cylinder is $1\frac{1}{2}$ in.; determine the angle of inclination of the thread and its length.

Ans. (1) $3^\circ\ 2'\ 12''$. (2) 339·7 in.

62. *The form of a screw with a square thread.*—In the last article was considered the form of the geometrical curve called the helix. If we suppose that instead of the triangle ACB we have a solid, such that, when it surrounds the cylinder, its upper face projects at right angles to the cylinder at every point, as shown in the annexed figure; this upper surface will be of the same form as that of the square-threaded screw; if now the lower part of this projection be cut away, so as to leave a protecting edge of uniform thickness, we shall obtain a screw with a square thread, as shown in fig. 64 *, a section of which made by a plane passing through the axi of the cylinder is shown in fig. 65. The student will remark that the thread of a screw, though a very common object, has a very remarkable form; for instance, the curve aa' (fig. 64), which when prolonged passes through the points aa_1, a_2 (fig. 65), is a helix, as also is the curve b'_1b' (fig. 64), which when prolonged will pass through the points $b\,b_1\,b_2$ (fig. 65). Also if the thread were cut by a cylinder with the same axis as that of the screw, and

Fig. 63.



* When there is a considerable distance between two consecutive turns of the thread, as is the case with the screw represented in the figure, it is usual to have a second intermediate thread running round the cylinder.

whose surface falls anywh ere between a and b (fig. 64.), the curve of section would be a helix, as indicated by the

Fig. 64.

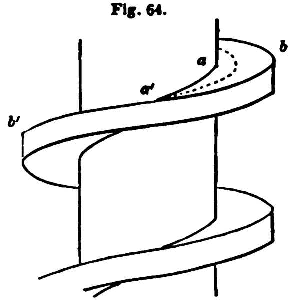
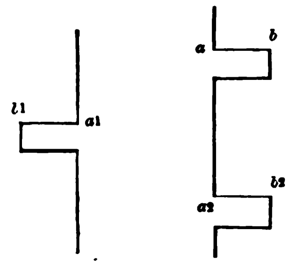
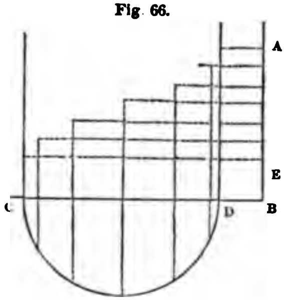


Fig. 65.



dotted line, the triangles whose hypotheneuses form these helices will all have the same height, viz. $a a_2$ or $b b_2$ (fig. 65), but their bases will be the circumferences of their respective cylinders. In order to gain a familiar acquaintance with the form of the square-threaded screw, the student is recommended to examine one carefully, and then to make a drawing of it by the ordinary method of projection; this he will be able to do very easily if he considers the following construction, which sufficiently exemplifies the rule for drawing the projection of a helix; draw AB and BC at right angles to each other; take C D, the diameter of the cylinder, and on it describe a semicircle; take A E, half the distance between two turns of the thread, and divide it into any even number (6) of equal parts, and divide the semi-circumference into the same number (6) of equal parts; through the points thus found on the circumference draw perpendiculars to C B, and cut them in order by

Fig. 66.



perpendiculars to A B drawn through the points of division of A E. The points of intersection thus obtained are situated on the required projection of the helix.

Ex. 282.—If h is the height between two turns of the thread of a screw, r and r_1 the radii of the external and internal cylinders, and θ and θ_1 the angles of inclination of the external and internal helices, show that

$$\tan(\theta_1 - \theta) = \frac{2\pi h (r - r_1)}{4\pi^2 r r_1 + h^2}$$

and show that the formula gives a correct result when $r_1 = 0$.

Ex. 283.—If the thread of the screw in Ex. 281 were cut half an inch deep determine the difference between the lengths of the interior and exterior helices, and the inclination of the mean helix.

Ans. (1) 112·8 in. (2) 3° 38′.

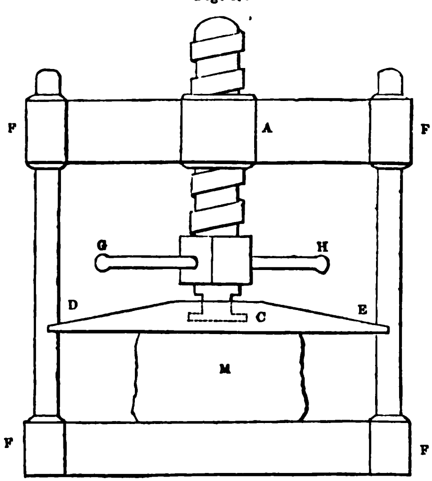
Ex. 284.—The external and internal radii of the thread of a square-threaded screw are r and r_1; its thickness (measured parallel to the axis) is a; show that the volume of one turn of the thread is $\pi (r^2 - r_1^2)\, a$.

Ex. 285.—A wrought iron screw is 1 ft. long, and $1\frac{1}{2}$ in. in radius, the thread makes 3 turns in 2 in., its thickness is $\frac{1}{3}$ in., its depth $\frac{1}{2}$ in., find its weight, and the weight of the part cut away when the screw was made.

Ans. (1) 276·1 oz. (2) 106·2 oz

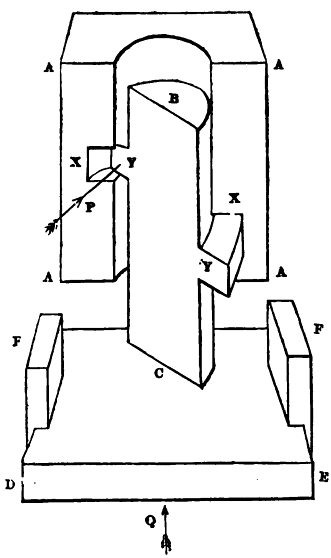
63. *The Screw Press.*— The most familiar application of the screw occurs in the case of the screw-press, and as it is very desirable that the student should get a clear conception of the mode of action of the forces in the case of the screw, he will do well to examine a screw-press; its most familiar form is represented in the annexed figure, and can be sufficiently described as follows: FFFF is a strong

Fig. 67.



frame; at A in the middle of the cross piece is a hollow nut, on whose interior surface is cut a groove, called the companion screw, which the thread of the screw BC exactly fits; the end C of the screw is fixed to the piece DE in such a manner that the screw is free to turn, while the piece DE can only move in a vertical direction in consequence of the guides FF, and FF; it moves downward when the screw is turned by the handle GH in one direction, and upward when the screw is turned in the opposite direction; in the former case a pressure is exerted on mass M which it is the purpose of the machine to compress. The action of the forces in this case will be understood by considering the annexed figure, in which AAAA represents a section of the nut, BC of the screw, FF the guides, DE the moveable piece, YY the thread of the screw, XX the groove of the companion; the pressure P is equivalent to the pressure at the end of the arm which tends to turn the screw; Q is the reaction against DE which balances P; the frictions called into play in this case are the following: (1) between the thread and the groove, (2) between the end of the screw and the piece DE, (3) between the guides F, F and the sides of the piece DE; (4) between the cylindrical surfaces B and A. It is not easy to obtain the relation between P and Q in the state bordering on motion when all the frictions are taken into account; the frictions marked

Fig. 68.



(3) and (4) are, however, small, and in the following pages will be omitted.

Ex. 286.—In Ex. 264 if α is the inclination of AB, and ϕ, ϕ_1, ϕ_2, the limiting angles of resistance between the surfaces at AB, HK and CD respectively, show that when P is on the point of moving forward

$$P = W \frac{\cos\phi_1 \sin(\alpha+\phi+\phi_2)}{\cos\phi_2 \cos(\alpha+\phi+\phi_1)}$$

Ex. 287.—Prove independently that, in the last example,

$$P = W \tan(\alpha+\phi)$$

when only the friction on AB is taken into account.

Ex. 288.—Show that in the case of the screw press the relation between P and Q is given by the formula

$$\frac{Pa}{r} = Q \tan(\alpha+\phi)$$

where a is the length of the arm on which P acts, r the radius of the screw, α the angle of inclination of the thread, and ϕ the limiting angle of resistance between the thread and groove; all other frictions being neglected.

Suppose the pressure Q to cause pressures $q_1, q_2, q_3, \ldots$ at different points of the thread of the screw, and suppose $p_1, p_2, p_3, \ldots$ to be the pressures which acting horizontally in directions touching the surface of the cylinder at those points would be on the point of overcoming $q_1, q_2, q_3 \ldots$ respectively, then the relation between p_1 and q_1 must be the same as that between P and W in Ex. 287. Hence

$$p_1 = q_1 \tan(\alpha+\phi)$$

and similarly
$$p_2 = q_2 \tan(\alpha+\phi)$$
$$p_3 = q_3 \tan(\alpha+\phi)$$

and therefore $p_1 + p_2 + p_3 + \ldots = (q\ \ + q_2 + q_3 + \ldots) \tan(\alpha+\phi)$.

Now $p_1, p_2, p_3, \ldots$ have the same tendency as P to turn the screw round its axis, and therefore the principle of moments gives us

$$Pa = p_1 r + p_2 r + p_3 r + \ldots$$

also since the pressures $q_1, q_2, q_3, \ldots$ are all parallel to Q's direction we have

$$Q = q_1 + q_2 + q_3 + \ldots$$

$$\therefore \frac{Pa}{r} = Q \tan(\alpha+\phi).$$

Ex. 289.—Show, by a method similar to that employed in the last example, that when all the frictions are neglected

$$Pa = Qr \tan\alpha$$

and that P : Q :: distance between two turns of the thread of the screw : the circumference of the circle described by the point at which P acts.

Ex. 290.—There is a screw with a square thread the radius of which is 1 in.; the distance between two turns of the thread is $\frac{1}{8}$ in., the nut is of cast iron and the screw of wrought iron, their surfaces are well greased, determine the pressure that would be produced on the substance in the press if we neglect all the frictions but that between the thread and the groove, when the screw is turned by a pressure of 150 lbs. acting at a distance of 3 ft. from the axis of the screw. *Ans.* 35,275 lbs.

Ex. 291.—In the last Example determine Q if the screw is not greased. *Ans.* 22,007 lbs.

Ex. 292.—Find the number of turns per foot which the thread of a perfectly smooth screw will make whose power is the same as that of the screw described in Ex. 290. *Ans.* $12\frac{1}{2}$ nearly.

Ex. 293.—If in any screw reckoned perfectly smooth a pressure P were required to compress a substance with a pressure Q, and if P′ were the additional pressure required in consequence of the friction between the thread and the groove, show that

$$P' = \frac{2\mu P}{\sin 2a}\text{. very nearly}$$

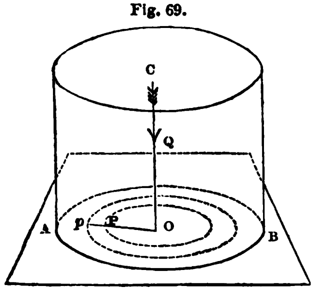
where a is the angle of inclination of the thread of the screw, and μ the coefficient of friction—neither being large.

Ex. 294.—If the screw described in Ex. 290 has to exert a pressure Q, find both from first principles and from the formula in the last example the value of $\frac{P'}{P}$. *Ans.* (1) 1·885. (2) 1·890.

Ex. 295.—The diameter of the screw of a vice is 1 in. and the thread makes 4 turns to the inch, the whole is of cast iron and the screw is well greased; the handle by which it is turned is 6 in. long and is urged by a pressure of 100 lbs.; the jaws of the vice hold an ungreased piece of wrought iron; find the pressure requisite to extract it. *Ans.* 2530 lbs.

64. *Friction on the end of the screw.*—Let ABC be a cylinder or pivot, the end of which is urged against a rough plane by a pressure Q acting along its axis OC, the cylinder is supposed to be on the point of turning round the axis, and is opposed by the friction, it is required to determine the moment of the frictions with respect to the axis OC.

Fig. 69.

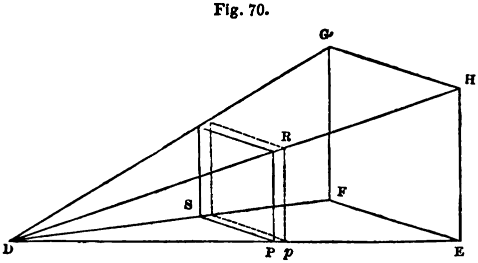


It may be assumed that the inequalities of the surfaces

will wear away, and that the pressure will be equally distributed; consequently if ρ is the radius of the pivot (say in inches), $\frac{Q}{\pi\rho^2}$ will be the pressure per square inch, and consequently $\frac{\mu Q}{\pi\rho^2}$ will be the friction per square inch; hence if we consider a small ring enclosed between two circles, whose radii OP and Op are respectively r and $r + \delta r$, its area will ultimately equal $2\pi r \delta r$, and the pressure on it will equal $\frac{2\mu Q}{\rho^2} r \delta r$. Now the friction at every point of this ring acts in a direction perpendicular to the radius at that point, and hence the sum of the moments of the frictions on this ring with respect to the axis will ultimately equal $\frac{2\mu Q}{\rho^2} r^2 \delta r$; the same will be true of any other ring, and therefore we shall obtain the required moment if we divide the area into a great number of rings, and ascertain the limit of the sum of the moments of the frictions on each ring; this can be done as follows:

Take $DE = \rho$ and at right angles to it draw $EF = \rho$, perpendicularly to both draw $EH = \frac{2\mu Q}{\rho}$, complete the

Fig. 70.



rectangle EFGH, and complete the pyramid DEFGH; take $DP = r$ and $Pp = \delta r$, and through P and p draw planes parallel to the base inclosing the lamina PRS, then

it is plain by similar triangles that $PS = r$ and $PR = \frac{2\mu Q}{\rho^2} r$, consequently the volume of the lamina is ultimately equal to $\frac{2\mu Q}{\rho^2} r^2 \delta r$, *i.e.* the moment of the friction on the ring is correctly represented by the volume of the lamina, and the same being true of any other lamina, we shall have the moment of the whole correctly represented by the volume of the pyramid*, *i.e.* the moment equals $\frac{1}{3} \rho \times \rho \times \frac{2\mu Q}{\rho}$ or moment of friction $= \frac{2}{3} \rho Q \mu$.

Ex. 296.—If the screw rests on a hollow pivot whose internal and external radii are respectively ρ_1 and ρ, show that the moment of the friction round the axis of the screw is given by the formula

$$\frac{2}{3} \cdot \frac{\rho^3 - \rho_1^3}{\rho^2 - \rho_1^2} \cdot Q\mu.$$

and show from this formula that when ρ_1 is very nearly equal to ρ the friction is very nearly equal to $\rho Q \mu$.

Ex. 297.—In the screw when the friction on the end as well as the friction on the thread is taken into account then

$$P = \frac{rQ}{a} \tan(\alpha + \phi) + \frac{2}{3} \cdot \frac{\rho}{a} Q\mu$$

where ρ is the radius of the *end* on which the screw rests.

[Referring to Ex. 288 the equation deduced from the principle of moments will become

$$Pa = rp_1 + rp_2 + rp_3 + \ldots + \tfrac{2}{3} \rho Q \mu].$$

Ex. 298.—It is required to compress a substance with a force of 10,000 lbs.; the screw with which this is done has a diameter of 3 in., and its thread makes 1 turn to the inch; the arm of the lever is 2 ft. long; determine the pressure P that would be required (1) if all frictions were neglected, (2) if the friction between the thread and groove, (3) taking also into account the friction on the end of the screw which is 1 in. in radius; the surfaces being iron on iron well greased.

Ans. (1) 66·3 lbs. (2) 129·6 lbs. (3) 157·5 lbs.

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* The student who understands the Integral Calculus will perceive that the above construction is equivalent to integrating the expresion $\frac{2 Q\mu}{\rho^2} r^2 dr$ between the limits of $r = o$ and $r = \rho$.

Ex. 299.—An iron screw 4 in. in diameter communicates motion to a nut, the force is applied at the extremity of a lever 1 ft. long; the inclination of the thread of the screw is 6°; determine the relation between the pressure applied and the weight raised by the nut, taking into account the frictions between the thread and groove, and the end of the screw whose diameter is 3 in.; the surfaces are cast iron: (1) when well greased, (2) when ungreased. *Ans.* (1) P=0·0427 Q. (2) P=0·0583 Q.

Ex. 300.—If the angle of the screw were 12°, the diameter of the screw and of its end 4 in., and the lever by which it is turned 2 ft. long, the surfaces being of cast iron and ungreased, what weight will a pressure of 1 cwt. overcome? *Ans.* 2730 lbs.

Ex. 301.—Determine the pressure required in Ex. 298 if the surfaces are of ungreased oak. *Ans.* 605 lbs.

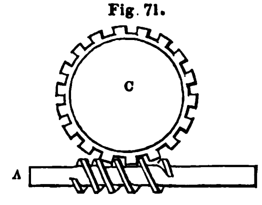
[The fibres may be reckoned parallel between thread and groove; the fibres of the screw of course rest endwise on the moveable piece.]

Ex. 302.—Given Q the pressure to be produced by the screw, r the radius of the mean thread, R the length of the arm, h the distance between two turns of the thread, μ the coefficient of friction between the thread and the groove, if the friction between the thread and the groove is the only one taken into account, show that the pressure to be applied at the end of the arm is given by the formula*

$$\frac{r}{R} \cdot \frac{h + 2\pi\mu r}{2\pi r - h\mu} Q.$$

65. *The Endless Screw.*—It is not very unusual to make a screw work with a toothed wheel; the arrangement of the pieces when this is done, will be sufficiently understood by an inspection of the annexed diagram; the screw AB may be mounted in a frame, and be turned by a winch; the teeth of the wheel (C) work with the worm of the screw, on turning which, the wheel is caused to revolve; as the screw has no forward motion, it will never go out of action with the wheel, and is, on that account, termed an *endless screw.* The reader will find in Mr. Willis's Principles of Mechanism† a discussion of the form that must be given

Fig. 71.



* This is the formula given in General Morin's Aide-Memoire, p. 309.

† p. 160.

to the teeth in order to secure equable working. When the machine is employed, it commonly happens that the screw drives the wheel; sometimes, however, the worm is driven by the wheel, as in the case of the fly of a musical box. In the former case it is easily shown, that if P is the pressure at the end of the arm which turns the screw, and Q that pressure exerted by the screw on the wheel in a direction parallel to the axis, then the relation between P and Q is the same as that determined in Ex. 288.

Ex. 303.—If a pressure P acting on the thread of a screw in a direction parallel to its axis is on the point of driving a pressure Q acting along a tangent to its base, show that

$$Q = P \tan (\alpha - \phi)$$

where α is the pitch of the screw at the working point, and ϕ the limiting angle of resistance between the driving and driven surfaces.

Ex. 304.—If the action of an endless screw is reciprocal, *i.e.* if it will act whether wheel or worm is driver, show that the pitch of the screw must be greater than ϕ and less than its complement.

Ex. 305.—An endless screw consists of a cylinder of cast iron the radius of whose base is 3 in.; the thread makes one turn in 4 in.; what is the greatest extent to which the thread can project if the tooth it drives is of cast iron and is ungreased? *Ans.* 0·98 in.

Ex. 306.—In the last example, if the depth of the thread be 1 in. what is the least distance between two turns of the thread with which the machine can work if the surfaces are greased? *Ans.* 2·513 in.

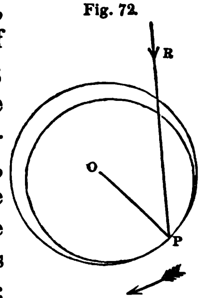
CHAP. VI.

OF THE EQUILIBRIUM OF BODIES RESTING ON AN AXLE, AND OF THE RIGIDITY OF ROPES; WHEEL AND AXLE, PULLEY.

SECTION I.

66. *Fundamental condition of equilibrium in the state bordering on motion, of a body capable of revolving round an axle.*—All the pressures acting on the body, can be reduced to a single resultant, to which, when the body is at rest, the reaction of the bearing must be equal and opposite; let the annexed figure represent the axle resting on its bearing; let R be the resultant of the pressures acting on the body, and let its direction cut the circumference of the bearing on point P; take O the centre of the bearing and join OP; this line is the normal to the point of contact;

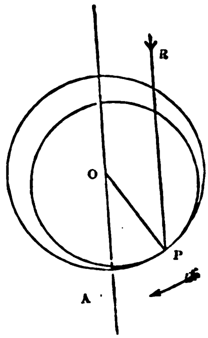
Fig. 72.



the body will therefore be in the state bordering on motion when the angle OPR equals the limiting angle of resistance, the motion being about to ensue in the direction indicated by the arrow head. This consideration enables us to give a very simple construction, which will apply to all cases in which the pressures act on the body in parallel directions. Take O the centre of the bearing, draw a line AO parallel to the directions of the pressures; if the body is about to move in the direction indicated by the arrow head, make the angle AOP equal to the limit-

ing angle of resistance, then a line RP parallel to OA

Fig. 73.



must be the direction of the resultant pressure, since this is the only line drawn parallel to OA which will cut the circumference in a point P such that the angle OPR equals the limiting angle of resistance; hence if we measure moments round P, we shall obtain the required relation between the pressures, the sum of those moments being equal to zero by art. 42. Of course if the motion is about to ensue in the contrary direction, the angle AOP must fall on the other side of OA. It will be remarked that the radii of the axle and its bearing are sensibly equal, so that though in the diagram they are represented as different that difference never enters the question.

67. *Friction of axles.*—When the body is in the state bordering on motion, the values of the coefficient of friction are the same as those given in the last chapter; the same is also true in cases of motion where no unguent is interposed; in nearly all cases of motion, however, an axle is kept well greased, both to prevent wear and to diminish the resistance; the unguent may be supplied at intervals, as in the case of a common cart wheel, or continuously as in the case of the wheel of a railway carriage; as might be expected a continuous supply of unguent is found to be the most effective means of diminishing the resistance; the following table gives the values of the coefficients of friction, and the limiting angle of resistance for the axles and bearings most commonly used; the coefficients of friction are taken from the experimental determinations of General Morin*, from which the limiting angle of resistance has

* Notions Fondamentales, p. 309. To avoid ambiguity the mean of some of Gen. Morin's results have been taken; thus, instead of 0·07 to 0·08, the above table gives 0·075.

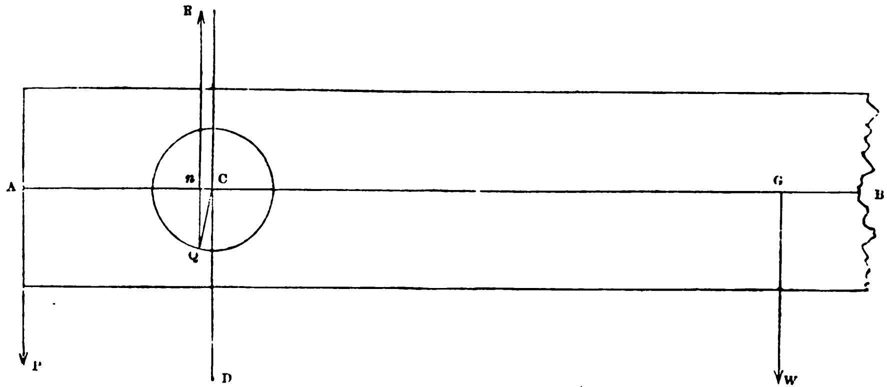


Fig. *e*. p. 127.

been calculated—those cases have been selected in which the unguent is most *effective* in diminishing friction.

TABLE XII.

FRICTION OF AXLES MOVING ON THEIR BEARINGS.

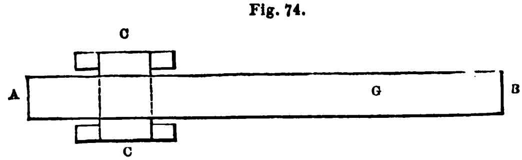
Axle and Bearings.	Unguents.	Renewed at intervals.		Renewed continuously.	
		μ tan ϕ or sin ϕ.	ϕ	μ tan ϕ or sin ϕ.	ϕ
Cast Iron on Cast Iron.	Oil of olives, tallow, hog's lard, soft gom	0·075 (mean)	4° 17′	0·054	3° 15′
Wrought Iron on Cast.	Do.	0·075 (mean)	4° 17′	0·054	3° 15′
Wrought Iron on Brass.	Oil of olives, tallow, hog's lard.	0·075 (mean)	4° 17′	0·054	3° 15′
Wrought Iron on Lignum-vitæ.	Oil, or hog's lard.	0·11	6° 16′		
Brass on Brass.	Do.	0·095 (mean)	5° 25′		
Brass on Cast Iron.	Oil or tallow.	. . . .	. .	0·0485 (mean)	2° 46′

Ex. 307.—Let AB (fig. *e*) be a beam moveable about a wrought iron axle which rests on a cast iron bearing, and whose axis passes at right angles through the axis of the beam *; the centre C of the axle is 12 in. from A, and 30 in. from the centre of gravity of the beam and axle; the radius of the axle being 3 in., the weight of the whole (*i. e.* of the beam and axle) is 400 lbs.; find the weight which when hung at A will just cause the end A to descend.

Draw the figure to scale; draw through C the vertical line CD, and make the angle DCQ equal to the limiting angle of resistance (10° 45′) draw the

* Of course there are in reality two bearings situated symmetrically with reference to the length of the beam, each of which supports half the united

Fig. 74.



pressures P and W; the *plan* of the machine being shown in the accompanying figure.

vertical line QR cutting AB in n; then this being the direction of the reaction the principle of moments gives us

$$P \times A\,n = W \times n\,G$$

but since n C is very small, it is desirable to construct the axle on a larger scale; this is done in fig. f, from which we obtain C n equal to 0·57 in.; hence we find P equal to 1069·8 lbs.; a result precisely the same as that obtained by calculation.

Ex. 308.—In the last Example determine the value of P which will just prevent the beam from falling when no unguent is used. *Ans.* 936·5 lbs.

Ex. 309.—Determine the magnitude and position of the resultant pressure in Ex. 307 if we suppose P = 1020 lbs.; and determine the magnitude of the angle its direction makes with the normal to the point of its application.

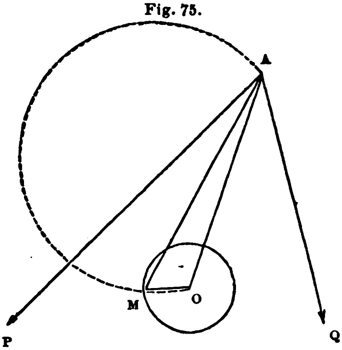
Ans. (1) 1420 lbs. (2) $Cn = \frac{12}{71}$ in. (3) $CQn = 3^\circ\ 13'\ 47''$.

Ex. 310.—There is a beam of oak AB whose length is 30 ft., depth 2 ft., and thickness 1 ft.; at right angles to its face passes an axle of wrought iron the part of which within the beam is 8 in. square, the projecting part on each side is 6 in. in diameter and 6 in. long (so that its total length is 2 ft.); its axis is situated 10 ft. from the end A, at which end is exerted a pressure of 5000 lbs.: find the pressure at B which will just keep the beam from turning, and the amount to which that pressure must be increased if it is on the point of overcoming the pressure at A; the axle rests in an oaken bearing ungreased. *Ans.* (1) 1540 lbs. (2) 1710 lbs.

Ex. 311.—If a string were wrapped round the grindstone described in Example 16, determine the greatest weight that could be tied to the end of the string without causing motion, supposing the bearing to be of cast iron well greased. *Ans.* 4·8 lbs.

68. *Conditions of equilibrium of two pressures acting in directions not parallel on a body capable of turning round an axle.*—Let P and Q be the two pressures acting along the lines AP, AQ; take O the centre of the axle, join AO, let the limiting angle of resistance between the axle and its bearings be ϕ, and suppose P to be on the point of preponderance; on the side of the line AO towards P describe a segment of a circle

Fig. 75.



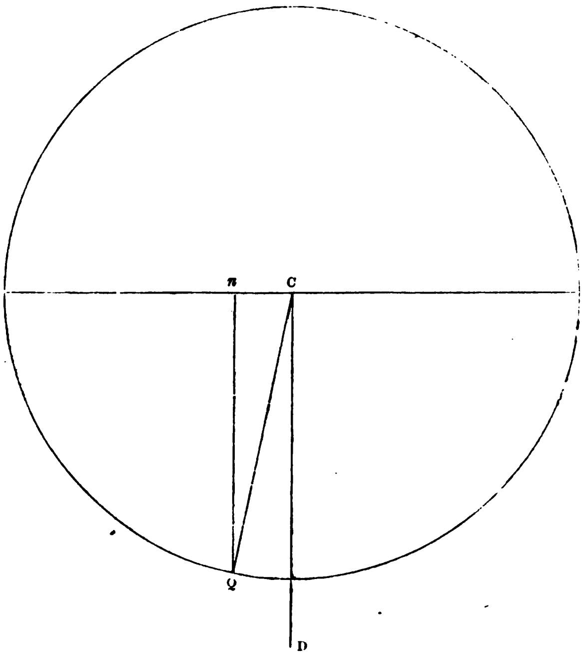
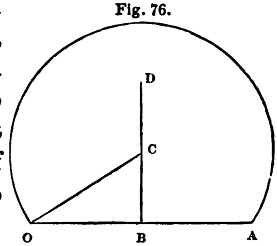


Fig. *f*. p. 128.

containing an angle ϕ (Eucl. 33—III.) cutting the circumference of the axle (or bearing) in M, join AM, then AM must be the direction of the resultant of P and Q; since this is the line which makes an angle $AMO = \phi$ with the normal to the point in which it cuts the circumference of the axle. The required condition is therefore that the moment of P round this point, shall equal the moment of Q round the same point. Of course if Q were the preponderating pressure, the segment of the circle must be drawn on the other side of OA.

It may be added, that the following construction is performed with somewhat greater facility than that given in (Eucl. 33—III.);—bisect OA in B, draw BD at right angles to OA, and on the side of the line required lay down the angle $AOC = 90° - \phi$ cutting the line BD in C, then it is evident that C is the centre of the circle of which the segment to be drawn is a part.*

Fig. 76.



Ex. 312.—In Ex. 307 suppose the pressure P to act perpendicularly to AB, determine the amount of P which will *just* keep the beam inclined at an angle of 45° to the horizon, no unguent being used to the axle.

Ans. 661 lbs.

Ex. 313.—In Ex. 311 suppose the grindstone to be turned by a pressure P that acts perpendicularly at the end of a winch $1\frac{1}{2}$ ft. long; find the magnitude of P which will just bring the grindstone into the state bordering on motion, (1) when the winch is vertically over the axis, (2) when it has turned through an angle of 45° from that position.

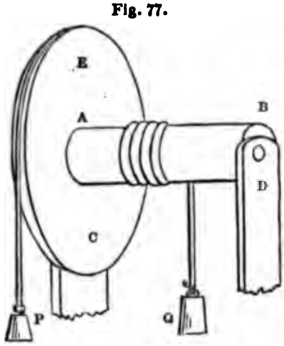
Ans. (1) 6·33 lbs. (2) 6·36 lbs.

69. *Wheel and Axle, Pulleys.*—The wheel and axle and the pulley are familiar examples of bodies capable of mov-

* In solving examples of bodies in equilibrium round an axle the figure must be drawn on a large scale; the student will probably find it convenient to make a separate figure for determining the lines which depend on the magnitude of the axle. See Ex. 307.

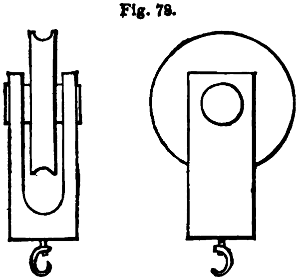
ing round a fixed axle; they may be sufficiently described as follows:

Fig. 77.



(1.) *The wheel and axle.*—Let AB represent a cylinder of wood or some other material called the axle, to the end of which is firmly fixed a cylinder of a large diameter EC called the wheel; they rest on a pair of bearings by means of a small cylindrical axis, one end of which is D, the geometrical axes of all these cylinders being coincident; ropes are wrapped round the wheel and axle respectively, to the ends of which weights P and Q are attached; if P is so large as to descend, it will do so by turning the machine; this will wind up Q's rope, and thereby cause that weight to ascend. It is usual to describe the wheel and axle in the above form, in order to give definiteness to the calculation; in practice, however, a winch commonly supplies the place of the wheel.

Fig. 78.



(2.) *The pulley* is simply a thin cylinder with a groove cut in its circumference, on which a rope can rest: the cylinder is capable of turning round an axis, which is supported by a piece called a block; this well known machine is represented in the accompanying diagram. When several pulleys are combined into a single machine, they constitute what is called a system of pulleys; the system most commonly used is called the tackle; it consists of two blocks containing pulleys (under these circumstances called sheaves) which are either equal

in number, or else the upper block contains one more sheave than the lower; the upper block is fixed, while the lower carries the weight; one end of the rope by which the weight is raised is fastened to one of the blocks, and passes in succession round each of the sheaves, as represented in the accompanying diagram: but it must be added that the sheaves in each block are commonly made equal to each other, and arranged one behind the other on a common axis. Another system of pulleys, called the Barton, is sometimes employed; it consists of one fixed and any number of moveable pulleys; to the block containing each moveable pulley is fastened a rope, which after passing under the next pulley (thereby supporting it) is fastened to a fixed beam. The last of these pulleys carries the weight to be raised; the rope which carries the first moveable pulley passes over the fixed pulley; on shortening this rope the pulleys, and with them the weight, are raised; the arrangement is shown in the accompanying diagram (Fig. 80); it rarely happens that more than one moveable pulley is employed.

Fig. 79.

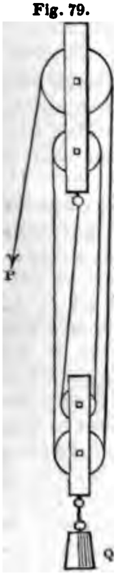
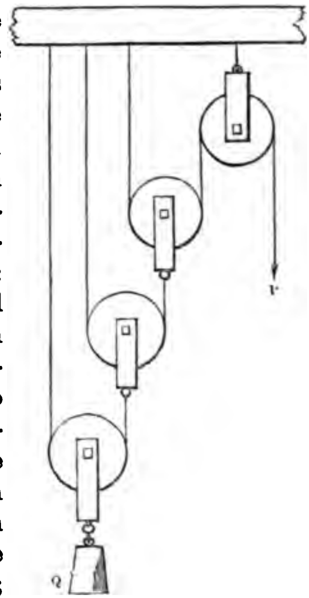


Fig. 80.



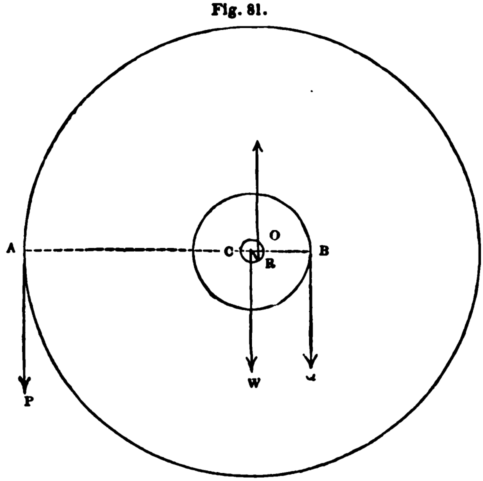
It is to be observed that the rigidity of the cords, *i.e.* their want of perfect flexibility, plays an important part in calculations concerning the mechanical power of the wheel

and axle, and of the pulley; we will first give some examples of the relations between P and Q when the rigidity of the cords is neglected, and will then explain the method of taking that resistance into account.

Ex. 314.—A wheel and axle weigh 1 cwt., the radius of the wheel is 2 ft., of the axle 6 in., the diameter of the axis 1 in., it is of wrought iron resting on a bearing of cast iron and is well greased; if Q equals 1000 lbs. find magnitude of P, (1) when it just supports Q, (2) when it is on the point of raising Q.

[With centre O and radii OA = 24 in., OB = 6 in., OC = $\frac{1}{2}$ in., describe circles and draw AOB horizontal; the pressure P may be conceived to act at A, W = 112 lbs. at O, and Q = 1000 lbs. at B; make the angle WOR equal to 5° 40′; the point R must be that about which the pressures P, W, and Q balance when P just supports Q; if P is on the point of overcoming Q the angle WOR must be constructed on the other side of OW.]

Fig. 81.



Ans. (1) 247·1 lbs.
(2) 253 lbs.

Ex. 315.—Determine the magnitude of the pressure P which would just bring the wheel and axle of the last Example into the state bordering on motion when the pressure Q is removed. *Ans.* 0·23 lbs.

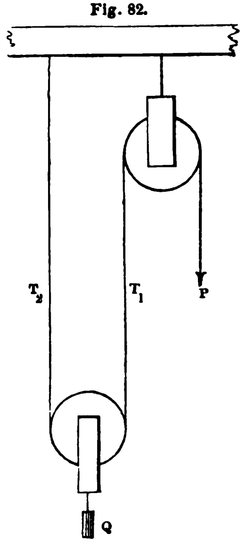
Ex. 316.—A fixed pulley 1 ft. in radius moves on an axle of wrought iron 1 in. in radius which turns in an oaken bearing, find the relation between P and Q in the state bordering on motion—P being the preponderating pressure—(1) neglecting the weight of the pulley, (2) supposing it to weigh 20 lbs.; the axle being without unguent, and P and Q acting vertically downward. *Ans.* (1) P = 1·092 Q. (2) P = 1·092 Q + 0·92 lbs.

Ex. 317.—Determine the relation between P and Q in the last case supposing P and Q to act vertically upward, *i. e.* in a direction *opposite* to that in which the weight of the pulley acts. *Ans.* P = 1·092 Q − 0·92 lbs.

Ex. 318.—In a Burton consisting of one fixed and one moveable pulley

whose dimensions and weights are the same as those assigned in Ex. 316, determine the relation between the power and the weight it is on the point of raising.

Fig. 82.



[Let the accompanying figure represent the pulleys, and let T_1, T_2 be the tensions on those portions of the cords against which they are written, then P is on the point of overcoming T_1 under the same circumstances as those assigned in the second case of Ex. 316. Hence:

$$P = 1{\cdot}092\ T_1 + 0{\cdot}92$$

T_1 is on the point of overcoming T_2 under the same circumstances as those assigned in Ex. 317. Hence:

$$T_1 = 1{\cdot}092\ T_2 - 0{\cdot}92$$

also T_1 and T_2 support Q*, so that

$$Q = T_1 + T_2$$

on eliminating T_1 and T_2 from these three equations we obtain the relation between P and Q.] *Ans.* $P = 0{\cdot}57\,Q + 0{\cdot}45$ lbs.

Ex. 319.—Solve the last Example on the supposition that the axles had been well greased.
$P = 0{\cdot}512\,Q + 0{\cdot}08$ lbs.

Ex. 320.—Determine the relation between P and Q in a tackle containing 3 sheaves each of which weighs 10 lbs., is 6 in. in radius and turns on an axle $1\frac{1}{2}$ in. in diameter, of wrought iron turning upon oak and well greased—the different portions of the rope being supposed parallel.

[If T_1, T_2, T_3 are the tensions on each parallel portion of the rope (see fig. 79) we shall obtain the following equations

$$\begin{aligned} P &= 1{\cdot}025\ T_1 + 0{\cdot}12 \\ T_1 &= 1{\cdot}025\ T_2 - 0{\cdot}12 \\ T_2 &= 1{\cdot}025\ T_3 + 0{\cdot}12 \\ T_1 + T_2 + T_3 &= Q] \end{aligned}$$

Ans. $P = 0{\cdot}350\,Q + 0{\cdot}08$.

Remark.—It will be remarked that the last term in the answers to Ex. 316, 317, 318, 319, 320, result from the influence of the weight of the pulley on the friction; the term is so small that we can ordinarily neglect it without sensible error.

---

* It will be remarked that 20 lbs. is the weight of the *pulley* without the *block* that contains it; Q is strictly speaking the weight of the lower *block* the *pulley* and the weight raised.

Ex. 321.—If we neglect the influence of the weight of each sheave on the friction, and the rigidity of the rope, show that in a tackle of n sheaves of the same dimensions as in the last Example $P = \dfrac{0 \cdot 025\,(1 \cdot 025)^n}{(1 \cdot 025)^n - 1}\,Q$, and determine the numerical value of the ratio when $n = 6$. *Ans.* $P = 0 \cdot 182\,Q$.

Ex. 322.—If all resistances are neglected, find the weight that can be raised by a pressure of 100 lbs. by means of the tackle of 6 sheaves described in the last Example, and also the weight that could be raised if the frictions on the axles are taken into account.

Ans. (1) Q = 600 lbs. (2) Q = 549 lbs.

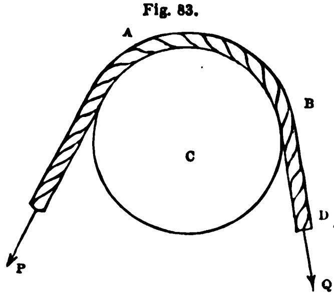
Ex. 323.—There is a fixed pulley the radius of which is 6 in., the diameter of its axle is 1 in., it is of wrought iron, and moves without an unguent in a bearing of cast iron; at one end of the rope is applied a pressure of 500 lbs. which is about to yield to a pressure P applied to the other end of the rope; if the parts of the rope when produced contain an angle of 60°, find P.

Ans. 514 lbs.

Ex. 324.—In the last Example determine the value of P which would *just* support the pressure of 500 lbs. *Ans.* 486·3 lbs.

70. *Rigidity of ropes.*—Let ABC represent a drum or pulley, moveable about an axis C, and let a rope ABD pass over it, to whose ends are applied pressures P and Q respectively, the friction of the rope being sufficient to prevent sliding; if one of the pressures P overcomes the other Q, it must be by causing the drum to revolve, thereby winding on the rope ABD; now the portion AB being circular, and BD being straight, the rope must be bent at the point B, and the rope not being perfectly flexible will offer a resistance to being thus bent, and a certain portion of the pressure P will be expended in overcoming the resistance. It is found that this rigidity of the rope can be taken account of by supposing Q to act along the axis of the rope, *i.e.* at a distance from C equal to $\frac{1}{2}$ of the sum of the diameters of the rope and

Fig. 83.



drum, and then increasing Q by a certain pressure; it is found by experiment that this additional pressure consists of a part depending purely on the rope, and another part proportional to Q; it is also found that, whenever other circumstances are the same, this additional pressure is greater as the curvature of the axis of the rope is greater, and therefore it can be correctly represented by the formula

$$\frac{A+BQ}{R}$$

where A and B are constants to be determined by experiment, and R is the effective radius of the drum, *i.e.* $\frac{1}{2}$ the sum of the diameters of rope and drum.

The principal experiments on the rigidity of ropes, are due to M. Coulomb*, whose results have been discussed by various writers. M. Morin considers that M. Coulomb's experiments are sufficient for the construction of empirical formulæ only in the cases of new dry ropes and of tarred ropes; from a discussion of the experiments† he obtains values of A and B which, after reduction, give the following values of the above formula:—

(1.) For new dry ropes, the resistance due to rigidity in lbs. equals

$$\frac{C^2}{R}\left\{0{\cdot}062994 + 0{\cdot}253868\, C^2 + 0{\cdot}034910\, Q\right\}.$$

(2.) For tarred ropes, the resistance due to rigidity in lbs. equals

$$\frac{C^2}{R}\left\{0{\cdot}222380 + 0{\cdot}185525\, C^2 + 0{\cdot}028917\, Q\right\}$$

where Q is estimated in lbs., C is the circumference of

* An abstract of Coulomb's Memoirs is given in Young's Nat. Phil. vol. ii. p. 171.

† Notions Fondamentales, pp. 316—330.

the rope in inches, and R the effective radius of the drum or pulley in inches. From these formulæ the following table has been calculated: —

TABLE XIII.
RIGIDITY OF ROPES.

Radius of Rope.	Circumf. of Rope.	New dry Ropes.		Tarred Ropes.	
		A.	B.	A.	B.
0·16 in.	1· in.	0·32	0·034910	0·41	0·028917
0·24	1·5	1·43	0·078543	1·44	0·065068
0·32	2	4·31	0·139640	3·86	0·115668
0·40	2·5	10·31	0·218183	8·64	0·180731
0·48	3	21·13	0·314190	17·03	0·260253
0·56	3·5	38·87	0·427643	30·56	0·354233
0·64	4	66·00	0·558560	51·05	0·462672
0·72	4·5	105·38	0·706723	80·03	0·585569
0·80	5	160·23	0·872750	121·50	0·722925

Rule.—Multiply B by Q in lbs., add the product to A, divide this sum by the effective radius of drum or pulley in inches, the quotient is the resistance in lbs.

If the resistance added to Q gives Q′, the relation between P and Q will be the same as that which obtains between P and Q′ acting by means of a perfectly flexible rope on a drum or pulley whose radius equals the effective radius.

It is to be remarked, that the resistance due to rigidity is only called into play whenthe rope is wound on to a drum; there is no resistance when the rope is wound off.

Ex. 325.—In Ex. 314 if the rope which supports Q is dry and 3 in. in circumference determine the addition to Q due to the rigidity of the rope, and determine the value of P, when it is on the point of raising Q.

Ans. (1) 51·8 lbs. (2) 286·8 lbs.

[The student must not overlook the increase in the effective radius of axle due to the thickness of the rope.]

Ex. 326.—In Ex. 316 determine the relation between P and Q in the first case, (*a*) supposing the rope to be 2 in. in circumference, (*b*) supposing the rope 4 in. in circumference.

(*a*) $P = 1{\cdot}102\,Q + 0{\cdot}36$ lbs. (*b*) $P = 1{\cdot}133\,Q + 5{\cdot}5$ lbs.

Ex. 327.—In Ex. 318 determine the relation between P and Q supposing

the ropes to be 2 in. in circumference, and the influence of the weights of the pulleys in increasing the frictions of the axles to be neglected.

Ans. $P = 0{\cdot}58\,Q + 0{\cdot}54$ lbs.

Ex. 328.—In Ex. 323 determine P, supposing the rope to be 3 in. in circumference. *Ans.* 541 lbs.

Ex. 329.—In Ex. 324 determine P, supposing the rope to be 3 in. in circumference. *Ans.* 461·6 lbs.

[The student must remember that now the rigidity of the rope assists P.]

71. *Cases in which axle is very small.*— If a body is capable of turning round an axle whose diameter is small compared with the distances at which the pressures that tend to turn the body act, it is usual to suppose the axle reduced to its geometrical axis; this supposition generally produces a great simplification in the relation between the pressures; but, of course, the simplicity involves a certain amount of inaccuracy. In the same manner it is very usual to solve mechanical questions on the supposition that the surfaces concerned are perfectly smooth and the ropes perfectly flexible. The following examples are, for the most part, to be solved on some one or more of these assumptions. The student is recommended, in every case, to consider the effect produced on the answer by the assumption from which it is obtained.

Ex. 330.—Determine the value of P in Ex. 307 on the supposition that the magnitude of the axle can be neglected. *Ans.* 1000 lbs.

[The student will remark that the variations of this question contained in Ex. 308 and 309 arise entirely from a consideration of the magnitude of the axle.]

Ex. 331.—There is a pole supported on two points 15 ft. apart; a weight of 3 cwts. is suspended at a distance of 6 ft. from one of these points, find the pressure it produces on each point. Also find the pressure if we suppose the rod to be a cylinder of wrought iron 16 ft. long, $1\frac{3}{4}$ in. in diameter, the ends being equally distant from the points of support.

Ans. (1) 201·6 lbs. and 134·4 lbs. (2) 266·6 lbs. and 199·4 lbs.

Ex. 332.—There is a wheel and axle, the radius of whose axis is supposed to be indefinitely small; the radius of the axle is 5 in. and that of the wheel $1\frac{1}{2}$ ft.; a weight of 300 lbs. acts at the circumference of the axle; find the pressure acting on the circumference of the wheel that will just balance it.

Ans. $83\frac{1}{3}$ lbs.

Ex. 333.—Let AB be a bar of wrought iron 16 ft. long, and resting with the end A on the ground, to the end B is tied a cord which is fastened to the ground at C, the length of BC and CA being 22 and 10 ft. respectively; at the point B is suspended a weight of 3 cwts., the rod weighs 10 lbs. per foot; find the tension on the cord; its circumference if it sustains the tension with perfect safety; and the coefficient of friction between the iron and the ground if the end is on the point of sliding.

Ans. (1) 399·5 lbs. (2) 3 in. (3) 0·376.

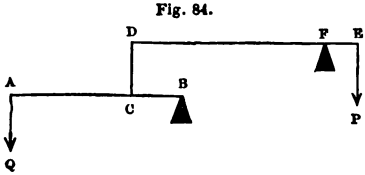
Ex. 334.—Let AB be a lever 16 ft. long moveable about a fulcrum D at a distance of 6 ft. from B, a weight of 28 lbs. is suspended from A and from B a weight of 336 lbs.; find the weight that must be hung at E (which is 7 ft. from D) to balance the lever. *Ans.* 248 lbs.

Ex. 335.—In the last Example suppose D to be centre of an axle of wrought iron $1\frac{1}{2}$ in. in diameter resting in a bearing of oak, suppose the rod to weigh 100 lbs. and its centre of gravity to be its middle point; find the weight which suspended at E will just keep the end B from falling, and also the weight which will just not cause the end A to fall.

Ans. (1) 216 lbs. (2) 223 lbs.

Ex. 336.—Let AB and DE be levers connected by a bar DC and capable of turning round fulcrums B and F; AB and DE are respectively 5 and 6 ft. long, AC is 3 ft., and FE is 9 in. long, the pressure P acting at E equals 1000 lbs. and is balanced by Q acting at A; find Q.

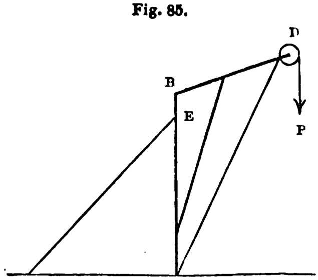
Fig. 84.



Ans. $57\frac{1}{7}$ lbs.

Ex. 337.—A crane CBD is sustained in a vertical position by the tension of a rope AE; its dimensions are as follows, BC, BD, BE and AC are respectively 19, 13, $1\frac{1}{2}$, and 16 ft. long, the angle CBD equals 108°; a weight P of 7 cwts. is supported by a rope that passes over a pulley D and is fastened to C; determine the tension on the rope AE, the weight of the crane and the dimensions of the pulley being neglected. *Ans.* 7·329 cwts.

Fig. 85.



Ex. 338.—In the last Example explain how the weight of the crane would affect the tension on the rope AE.

Ex. 339.—In Ex. 337 suppose the centre of the axle of the pulley is at D, the diameter of the pulley is 1 ft., that of its axle 2 in.; the surfaces being wrought iron on cast well greased; suppose the rope which sustains the

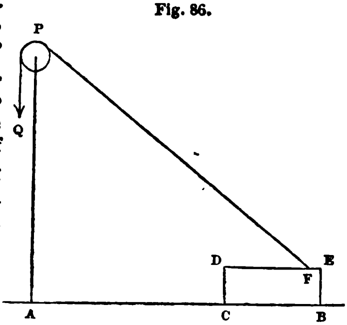
weight P is pulled in a direction parallel to AE with a pressure that is on the point of raising P; find the tension on the rope AE.

Ans. 3·41 cwts.

Ex. 340.—Let BCDE represent a block of Portland stone whose dimensions are 5 ft. long, 2 ft. high, and 2½ft. wide; a rope FPQ is attached to it, which after passing over a pulley P is pulled vertically downward by a pressure Q, which is just sufficient to raise the block: determine Q on the supposition that the dimensions of the pulley can be neglected, having given that EF equals 6 in. and BA and AP respectively 15 and 13 ft. the point A being vertically under P.

Ans. 1942 lbs.

Fig. 86.



Ex. 341.—If in the last Example the floor AB were of brickwork would the mass slide in the process of being raised? *Ans.* Yes.

Ex. 342.—If we suppose the pulley to have the same dimensions, &c. as that in Ex. 339, its centre being at P, determine Q. *Ans.* 1971 lbs.

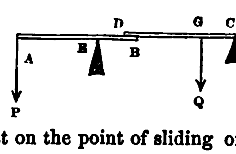
Ex. 343.—If in Ex. 340 we suppose the rope to have the dimension requisite for sustaining the tension on PF with perfect safety, determine the circumference of the rope. *Ans.* 6½ in.

Ex. 344.—Let AB represent a cylindrical bar of wrought iron 12 ft. long and 2 in. in diameter capable of turning round a fulcrum at B; at a point C, which is 15 ft. vertically over B, is placed a pulley the dimensions of which are neglected; to the end A of the bar is attached a rope which passes over C and sustains a weight P; on the bar is placed a weight Q of 3 cwts. at a distance of 3 ft. from A; find the magnitude of P which will just keep AB in a horizontal position. *Ans.* 404·25 lbs.

Ex. 345.—Let AB and DC be two levers of oak each 6 ft. long, capable of turning round fulcrums E and C respectively; EB equals 2 ft., the levers overlap 6 in., from A is suspended a weight P of 1 cwt.; find the weight Q which when suspended from G, at a distance of 2 ft. from C, can be just lifted by P.

Ans. 616 lbs.

Fig. 87.



[It must be remembered that the end B is just on the point of sliding on the underside of DC.]

Ex. 346.—In the last Example determine the magnitude of Q which will just allow the levers to disengage. *Ans.* 488·5 lbs.

[In this case P must be able to bring Q into the state bordering on motion when B has come to be just under D.]

Ex. 347.—In Ex. 345 determine the magnitude of Q if P will only just support it. *Ans.* 896lbs.

[In this case D is just on the point of sliding upon the upper side of AB.]

Ex. 348.—There is a circular table 6ft. in diameter supported on three legs which are under the points A, B, C; the triangle formed by joining ABC is equilateral, the point A is on the circumference, and B and C are each 1ft. from the nearest points of the circumference; the weight of the table is 56lbs.; find the pressure on each leg.

Ans. 6·5lbs. on A, 24·25lbs. on B, and 24·25lbs. on C.

Ex. 349.—In the last Example if ABC are all on the circumference, the sides AB, AC equal, and the angle BAC equals 30°; find the pressure sustained by each leg.

Ans. 26lbs. on A, 15lbs. on B, and 15lbs. on C very nearly.

Ex. 350.—A ladder 60ft. long, which weighs 7cwts. and balances on a point situated at a distance of 20ft. from its foot, has to be raised by means of a rope attached to the end of it and passing into a window whose height is 50ft. above the pavement; when the foot of the ladder is 8ft. from the wall of the house, and the length of rope from the window to the ladder is 20ft., find the tension on the rope and the angle the pressure on the foot of the ladder makes with the vertical. *Ans.* (1) 0·474cwts. (2) 3° 33′

72. *The Steel-yard.*—If a beam AB is suspended about a fine axis passing through its centre of gravity (G), and on the arm BG is placed a moveable weight W, then if a substance equal in weight to W is suspended from A, the beam will balance when W is at a distance from G equal to AG; if the substance equals twice the weight of W, the beam will balance when W's distance from G equals twice AG; and so on in any proportion. Hence, if the beam is made heavy at the end A, so that G is very near that point; the arm BG can be divided into *equal* divisions which shall indicate the weight of a substance suspended at A by means of the *position* occupied by W when it balances that substance. An instrument constructed on this principle is called a steel-yard, and is used when heavy substances have to be weighed, and extreme accuracy is not required; the advantage it possesses arises from the fact that the weights employed are much less heavy than the substance to be weighed. A very common application

of the principle of the steel-yard can be seen in the weighing machines employed at most railway stations.

Ex. 351.— Show that the graduations of the steel-yard must be equal even if the centre of gravity of the beam do not coincide with the axis; but that the graduations must begin from that point at which the moveable weight would hold the beam in a horizontal position.

SECTION II.

73. *Notation.* — Throughout the present section the letters ρ and ϕ stand respectively for the radius of the axle and the limiting angle of resistance between the axle and its bearing; also P and Q are the pressures in equilibrium, p and q are respectively the perpendiculars let fall on their directions from the centre of the axle; the letters A and B have the same meaning as in Art. 70. The student who has thoroughly mastered the first section of the present chapter, will find that in most of the following examples there is no difficulty except what arises from the fact of *algebraical* results being required instead of arithmetical.

Ex. 352.—When two parallel pressures P and Q acting towards the same parts balance round an axle, show that when P is on the point of overcoming Q

$$P\,(p - \rho \operatorname{Sin} \phi) = Q\,(q + \rho \operatorname{Sin} \phi).$$

How will this formula be changed if P and Q act towards contrary parts?

Ex. 353.—If P and Q are two parallel pressures and P is on the point of drawing up Q over a pulley whose effective radius is r, and weight W, show that

$$P\,(r - \rho \operatorname{Sin} \phi) = Q\,(r + \rho \operatorname{Sin} \phi) \pm W \rho \operatorname{Sin} \phi$$

where the positive sign is used if P and Q act downward, and the negative sign if they act upward; and that when the rigidity of the rope is taken into account the formula becomes

$$P\,(r - \rho \operatorname{Sin} \phi) = Q\left(1 + \frac{B}{r}\right)(r + \rho \operatorname{Sin} \phi) + \frac{A}{r}(r + \rho \operatorname{Sin} \phi) \pm W \rho \operatorname{Sin} \phi$$

74. *Remark.*—We have already seen in particular cases that the influence of the weight of the pulley on the friction of the axle is very small, as indeed is evident from the above formula, since W is commonly small compared with P and Q, and ρ sin ϕ is always small compared with r; now if we omit the last term it is evident that the formula will be the same whether P and Q act vertically upward or vertically downward, and can be written:

$$P = aQ + b$$

where a and b are written instead of the complicated expressions,

$$a = \left(1 + \frac{B}{r}\right) \cdot \frac{r + \rho \sin \phi}{r - \rho \sin \phi} \text{ and } b = \frac{A}{r} \cdot \frac{r + \rho \sin \phi}{r - \rho \sin \phi}$$

In the following questions a and b will have these values, and it will be understood in every question relating to combinations of pulleys that the effect of the weight on the friction of the axle is neglected; it must also be remembered that this is not the same thing as neglecting the weight entirely.

Ex. 354.—If P is on the point of lifting Q by means of a tackle in which each block contains one sheaf, show that

$$P = \frac{a^2}{1+a} . Q + b . (1 + \frac{a}{1+a}).$$

[See Examples 318 and 320.]

Ex. 355.—Obtain a formula similar to that contained in the last Example for the case in which there are in all three sheaves, and show that when there are n sheaves—all the ropes being considered parallel, we have

$$P = Q \frac{a^n (a-1)}{a^n - 1} + \frac{n b a^n}{a^n - 1} - \frac{b}{a-1}.$$

Ex. 356.—Show from the formula in the last example, and also from first principles, that when the passive resistances are neglected $P = \frac{Q}{n}$.

Ex. 357.—There is a block and tackle consisting of 6 sheaves each 3 in. in radius, whose axles are $\frac{1}{2}$ in. in radius and are of wrought iron turning on cast iron; the rope used is untarred and is 4 in. in circumference, the total

weight raised (*i.e.* the mass and lower block) is 1000 lbs.; find the pressure required, (1) taking into account the passive resistances, (2) neglecting them. *Ans.* (1) 390 lbs. (2) 133 lbs.

Fig. 88.

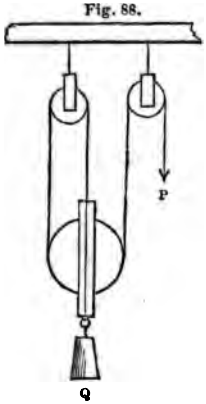
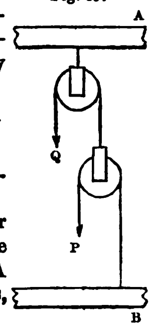


Fig. 89.



Ex. 358.—When the pulleys are arranged as in the annexed diagram (Fig. 88) show that the relation between P and Q is given by the following formula

$$P(1+a+aa_1)=a^2a_1Q+b(1+a+2aa_1)+ab_1(1+a).$$

where a, b, refer to the smaller pulleys and a_1, b_1 to the large pulley.

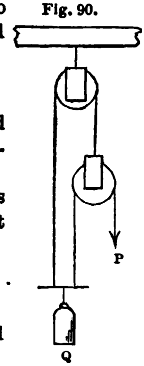
Ex. 359.—If a pair of similar pulleys are arranged as shown in the accompanying figure (89), where A and B represent immoveable beams, show that

$$P=\frac{a^2Q}{a+1}+b-\frac{aw}{a+1}.$$

where w is the weight of the moveable pulley?

Ex. 360.—In the last Example suppose each pulley to be similar to that described in Ex. 357, and the moveable pulley with its block to weigh 50 lbs.; the rope being dry and 4 in. in circumference, find the pressure required to raise a weight Q of 1000 lbs. and determine the corresponding values of P when the passive resistances are neglected. *Ans.* (1) 658 lbs. (2) 475 lbs.

Fig. 90.



Ex. 361.—If a pair of similar pulleys is employed to raise a weight Q in the manner indicated by the annexed diagram (Fig. 90); show that

$$(2a+1)P=a^2Q+b(2a+1)-aw$$

and determine P when Q weighs 1000 lbs., the pulleys and ropes being the same as in Ex. 360; and when passive resistances are neglected. *Ans.* (1) 432 lbs. (2) 317 lbs.

Ex. 362.—In the case of a tackle with three equal sheaves show that the pressure P which will just support a weight Q is given by the formula

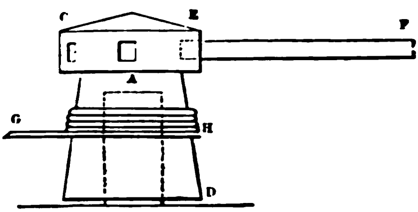
$$P=\frac{(a-1)Q}{a(a^3-1)}+\frac{3b}{a(a^3-1)}-\frac{b}{a-1}$$

and show that when the passive resistances are neglected equation reduces to $3P=Q$.

75. *The Capstan.*—This machine in one of its commonest forms consists of a cylindrical mass of wood, CD,

along the axis of which is cut a cylindrical aperture, which receives an axis AB (commonly of metal) on the top of which it rests; in the upper part of the capstan holes are cut, into which are inserted arms, such as EF, by means of which the capstan is turned, thereby winding up the rope GH which carries the weight.

Fig. 91.



Ex. 363.—In a capstan turned by two equal parallel pressures P acting towards opposite parts at equal distances a from the geometrical axis of the figure, let b be the radius of the cylinder round which the rope is wrapped, r the radius of the metal axle, μ_1 the coefficient of friction between the top of the axle and the capstan, and μ or $\tan\phi$ that between the side of the axle and the capstan, show that when the friction on the top of the axle is neglected

$$2\mathrm{P}a = (b + r\sin\phi)\left(\mathrm{Q} + \frac{\mathrm{A}+\mathrm{B}\,\mathrm{Q}}{b}\right)$$

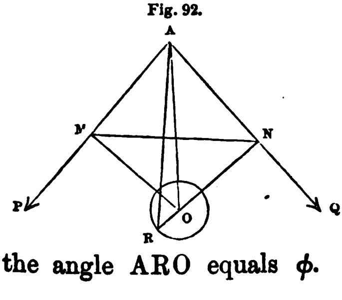
and when the friction on the top of the axle is taken into account

$$2\mathrm{P}a = (b + r\sin\phi)\left(\mathrm{Q} + \frac{\mathrm{A}+\mathrm{BQ}}{b}\right) + \tfrac{2}{3}\, r\mu_1 \mathrm{W}$$

where W is the weight of the capstan.

76. *Equilibrium of two pressures acting in given directions on a body capable of turning an axle.*—A geometrical construction has been already given by which the relation between the pressures can be determined; in the formula proved below the following notation is employed. Let P and Q be the pressures whose directions meet in A; P is on the point of preponderance; O the centre of the axle; draw the line AR so that the angle ARO equals ϕ. Let fall OM, ON perpendi-

Fig. 92.



culars on the directions of P and Q and join MN, then $PAO = \alpha$, $QAO = \beta$, $RAO = \theta$, $OM = p$, $ON = q$, $MN = L$.

Ex. 364.—In the above figure show that $L = p \cos \beta + q \cos \alpha$.

Ex. 365.—When P is on the point of overcoming Q show that $Pp = Qq\left(1 + \frac{L\rho \sin \phi}{pq}\right)$ very nearly.

[We have $P \sin(\alpha - \theta) = Q \sin(\beta + \theta)$

or since θ is a very small angle, this is very nearly equivalent to the following

$$P(\sin \alpha - \sin \theta \cos \alpha) = Q(\sin \beta + \sin \theta \cos \beta).$$

Remembering that $\sin \theta = \frac{\rho \sin \phi}{AO}$, and that $AO \sin \alpha = p$ and $AO \sin \beta = q$, we obtain

$$P(p - \rho \sin \phi \cos \alpha) = Q(q + \rho \sin \phi \cos \beta)$$

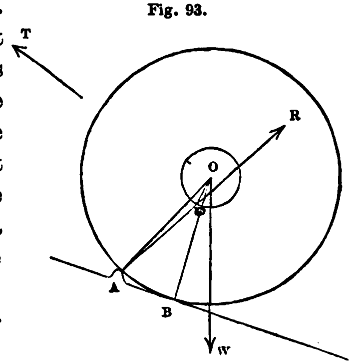
whence the approximate equation in the question is at once derived.]

Ex. 366.—Determine the answer to Ex. 312 by means of the formula of Ex. 365. *Ans.* 661 lbs.

Ex. 367.—Explain why the formula in Ex. 365 will not enable us to solve Ex. 313.

77. *The two-wheeled carriage.*—In this case we may consider that the weight of the carriage is equally distributed upon each wheel.

Fig. 93.



Now it will be observed that at each instant the wheel is lifted over a small obstacle A; then if O is the centre of the axle, and B the point of contact with the road, the angle A O B must have a certain magnitude, which we will denote by the letter γ. We will also denote the inclination of the road by α, and the angle between the direction of the traction and the road by β. Then the pressures concerned are, the trac-

tion T, the weight W, and the reaction R, of the point A, which, when T is on the point of moving W, must cut the circumference of the axle in a point D, such that O D R $= \phi$; then if we denote the angle O A R by θ, the relation between T and W will be easily obtained by the triangle of pressures.

Ex. 368.—When the wheel as above explained is on the point of moving, show that

$$T = W \frac{\sin(\alpha + \gamma + \theta)}{\cos(\beta - \gamma - \theta)}.$$

Ex. 369.—If A is the length of the arc AB, r and ρ the radii of the wheel and axle respectively, and if the road and the direction of traction are horizontal, show that

$$rT = W(A + \rho\phi) \text{ very nearly.}$$

Remark.—It appears from the experiments of General Morin that the traction is sensibly proportional to the weight directly and the radius of the wheel inversely, when the roads are paved or hard macadamized, and both the road and direction of traction are horizontal*; consequently it appears that for such roads, under the circumstances assigned in Ex. 369, the traction, as found by experiment, equals $\frac{kW}{r}$, where k is a constant quantity; but from the example it appears that $k = A + \rho\phi$, and hence the length of the arc A must be very nearly the same for the same road whatever be the radius of the wheel.

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* Morin, Notions Fondamentals, p. 353. The account of the carriage wheel given in the text is taken from Mr. Moseley's Mechanical Principles of Engineering, pp. 395, 6. 7. The general results of M. Morin's experiments will be found in the Appendix to Mr. Moseley's work. The reader will find a great deal of condensed information on the subject of carriage wheels in Dr. Young's Natural Philosophy, Lecture 18.

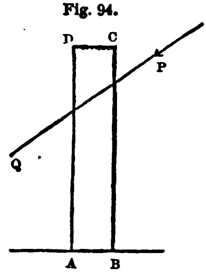
CHAP. VII.

ON THE STABILITY OF WALLS.

Section I.

78. *Conditions of the equilibrium of a wall.*—The questions that concern the equilibrium of walls fall into two classes. In the first place, it may be asked what is the pressure acting in a certain direction that can be supported by a given wall. In the second place, it may be asked what must be the dimensions of a wall to sustain a given pressure. Questions of the first class admit of an easy answer by construction; those of the second commonly require the use of more advanced mathematics. The first section of the present chapter will therefore contain questions that belong to the first of these classes.

Fig. 94.



Let A B C D represent the section of a wall, the base A B being on the level of the ground; let it be acted on by a pressure P along the line P Q: now it is considered that a wall, to be stable, must be capable of standing irrespectively of the adhesion of the mortar *; hence, if we suppose B D to be a continuous mass, and simply to *rest* on the section A B, and determine the pressure P which will be in the

* "Though ordinary mortar sometimes attains in the course of years a tenacity equal to that of limestone, yet, when fresh, its tenacity is too small to be relied on in practice as a means of resisting tension at the joints of the structure; so that a structure of masonry or brickwork, requiring, as it does, to possess stability while the mortar is fresh, ought to be designed on the supposition that the joints have no appreciable tenacity." Rankine, Applied Mechanics, p. 227.

act of turning the mass round the point A, this will be the greatest pressure that the wall can support, and can be determined by the rule that its moment with reference to the point A equals the moment of the weight of the wall with reference to the same point.

Ex. 370.—A wall of brickwork 2 feet thick and 25 feet high sustains on the inner edge of its summit a certain pressure on every foot of its length; the direction of this pressure is inclined to the horizon at an angle of 60°; find its amount when it will just not overthrow the wall. (See fig. *g*.)

Draw the section of the wall ABC to scale; make the angle BAN equal to 30°, then the pressure P acts along the line PN; draw CN perpendicular to PN; through G, the centre of gravity, draw the vertical line GM, cutting CB in M; the principle of moments gives us

$$P \times CN = W \times CM.$$

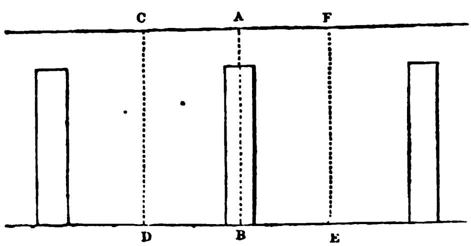
The weight W equals 5600 lbs.; CM equals 1 foot; CN, as obtained by measurement, equals 10·8 feet; whence P equals 518 lbs. When P is found by calculation it equals 520 lbs.

Ex. 371.—In the last example suppose the pressure to be applied by means of a bracket, at a horizontal distance of 3 ft. from the inner edge of the summit; determine its amount when it will just not overthrow the wall.

Ans. 685 lbs.

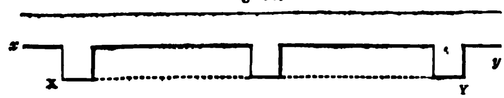
79. *The effect of buttresses.*—Let Fig. 95 represent the elevation of a wall, Fig. 96 its plan, and Fig. 97 its section made along the line AB; if now we neglect the weight of the buttresses, their effect in supporting the wall will be understood by inspecting in Fig.

Fig. 95.



96; for it is manifest that the wall would fall by being caused to turn round the line XY; but, if the buttresses were removed, by being caused to turn round the line

Fig. 96.



$x\,y$; so that, in the former case, the moments must be

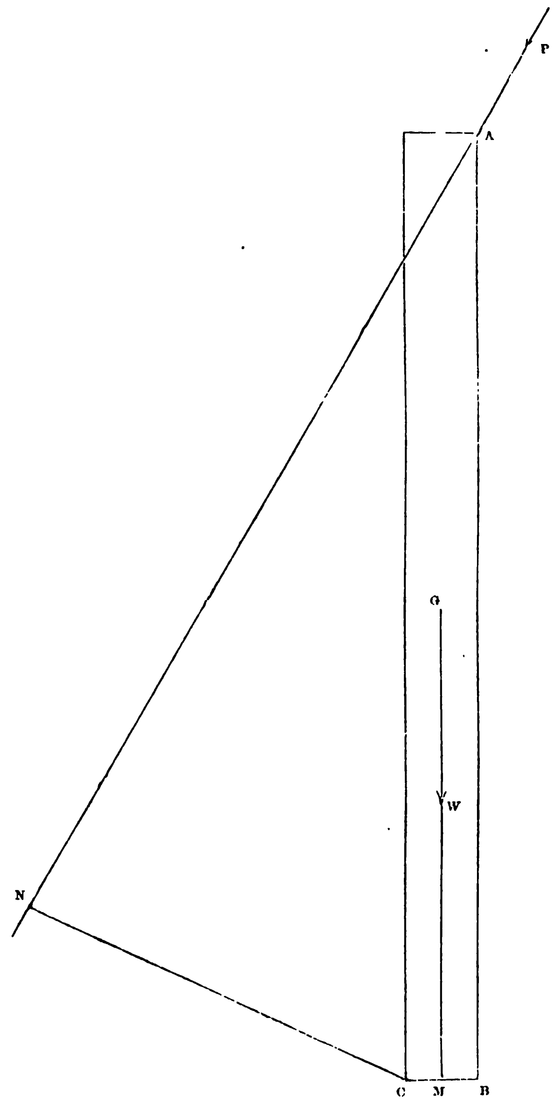
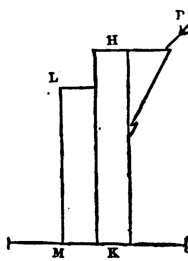


Fig. g. p. 148.

measured round M, in the latter round K: in other words, the introduction of buttresses diminishes the moment of P, and increases that of the weight of the wall. Their useful effect is still further increased by the fact that if the moment of the weight of the buttress is taken into account, it increases the moment of the weight of the wall.

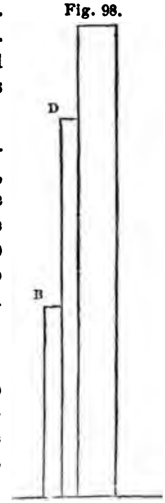
Fig. 97.



It is to be observed that if C D and E F be drawn at equal distances from A B, and at a distance from each other equal to the distance between the centres of two consecutive buttresses, then we may consider that the total pressure on C F is supported by the weight of the portion of the wall between C D and E F, and by the weight of the buttress.

Ex. 372. - In the last example if the wall were supported by buttresses 2 ft. thick*, to what can the pressure on each foot of the length of the wall be increased without overthrowing it:—the weight of the buttresses being neglected? *Ans.* 2609 lbs.

Fig. 98.



Ex. 373.—In Ex. 370 suppose the wall to be supported by counterforts reaching to the top of the wall, 1 foot thick, 1 foot wide, and 10 feet apart from centre to centre, determine the pressure on each foot of the length of the wall that can be supported; (1) when the direction of the pressure is inclined at an angle of 60° to the horizon; (2) when the direction is inclined at an angle of 30° to the horizon.

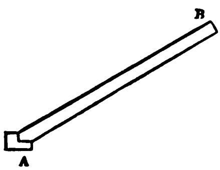
Ans. (1) 1145 lbs. (2) 562·8 lbs.

Ex. 374.—In each case of the last example determine to what the pressure can be increased if the buttress assumes the form of a Gothic buttress, as indicated in the annexed diagram, where A C and C E are each a foot square, and C D and AB are respectively 20 and 10 ft. high? *Ans.* (1) 1903 lbs. (2) 875 lbs.

* The *thickness* of a pier or buttress is measured in a direction perpendicular to the face of the wall.

80. *The thrust of props.*—Let A B represent a beam or prop resting on a fixed support at the end A; and suppose it to be acted on by certain pressures which are balanced by the reaction of the end A. That part of the reaction which acts along the axis of the beam A B is called the thrust of the prop, and is, of course, equal to the thrust produced by the pressures on the prop, the two being equal and opposite. If no pressure acts on the beam except at the end B, it is plain that the whole reaction from A must pass along the beam. For the following questions, which concern the thrust of props, it will be assumed that the *thickness* of props and beams can be neglected, except so far as it affects their weight.

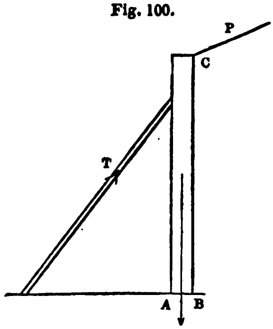
Fig. 99.



Ex. 375.—A wall of brickwork, 25 ft. high and 2 ft. thick, sustains on the inner edge of its summit a pressure of 1000 lbs. on every foot of its length, whose direction is inclined at an angle of 65° to the vertical; it is supported at every 5 ft. of its length by a prop 25 ft. long, resting against a point 3 ft. from the top; determine the thrust on the prop.

Ans. 7700 lbs.

Fig. 100.



[If the annexed figure represent a section of the wall and prop, the pressures acting are P, the pressure on the summit of the wall, W, its weight; these are balanced by T, the thrust of the prop, and the reaction of the ground AB: now, unless the prop is wedged up against the wall, it will not supply more pressure than is *just* sufficient to support the wall; consequently the resultant of P, W, and T must pass through A, at which point it will be balanced by the reaction of the ground; hence by measuring moments round A we can find T.]

Ex. 376.—In the last example determine the magnitude and direction of the pressure on the point A, without assuming a knowledge of the magnitude of T. *Ans.* 20900 lbs.

[Determine by construction the resultant of P and W, and the point in which its direction cuts that of T; the pressure in question will pass through this point and through A, so that its position is completely known; and then

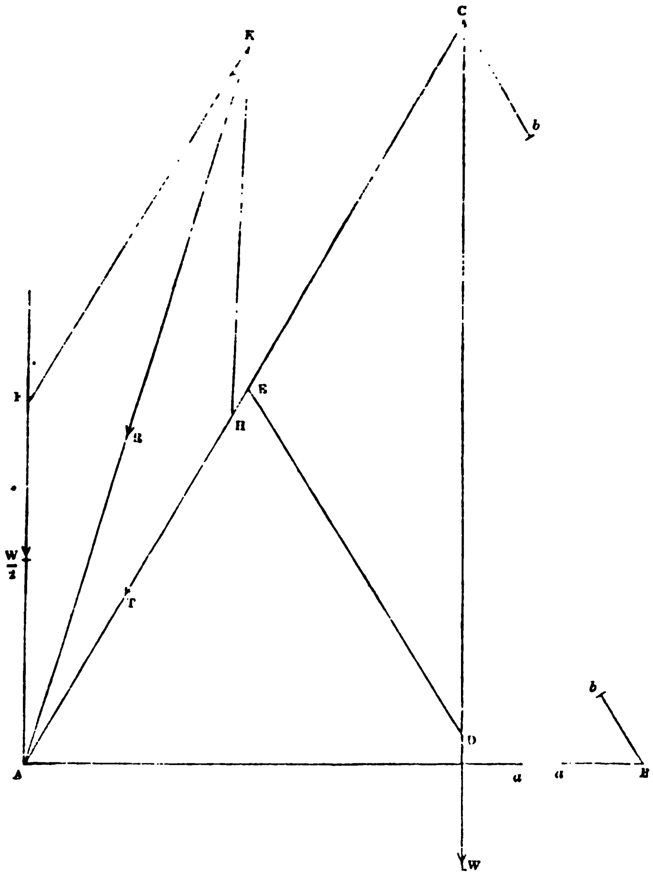
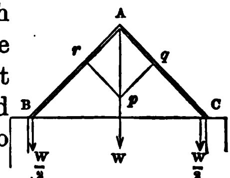


Fig. h. p. 151.

by measuring moments round any point in T's direction (*e. g.* the foot of the prop) its magnitude is at once determined.]

81. *The thrust produced by a roof on the wall which supports it.*—Let A B, A C represent two of the principal rafters of an isosceles roof, and let the whole weight sustained by each rafter (including its own weight) be represented by W: this weight will act at the middle point of the rafter, and can be replaced by weights equal to $\frac{W}{2}$ acting at each end of the rafter, so

Fig. 101.



that the total weight sustained by both rafters can be conceived to be distributed as shown in the figure, viz., W acting at A, $\frac{W}{2}$ at B, and $\frac{W}{2}$ at C; now the pressure W can only be sustained by the thrusts of the rafters; if then we take A p to represent W, and complete the parallelogram A q p r, the lines A q, A r will represent the pressures along A B and A C necessary to support W, and therefore also will be proportional to the thrust T produced on the walls along the lines A B and A C, the total pressure on each wall being the resultant of T and $\frac{W}{2}$. When the determination of the thrust is made for the purpose of ascertaining whether a certain wall will support the roof, it is much easier to regard the wall as acted on by two pressures T and $\frac{W}{2}$ than to regard it as acted on by their resultant.

Ex. 377.—There is a roof weighing 25 lbs. per square foot, the pitch of which is 60°; the distance between the side walls is 30 ft.; determine the magnitude and direction of the pressure on the foot of each rafter, the rafters being 5 ft. apart. (See fig. *h.*)

Let ABC represent the roof; then the weight (W) supported on each rafter equals 3750 lbs.; hence, when the weight is distributed, we have W at

C, $\frac{W}{2}$ at A, and $\frac{W}{2}$ at B; draw CW vertical, and take CD to represent 3750 lbs.; draw DE parallel to BC [which is broken in the figure as indicated by the letters a, a and b, b]; then CE represents the thrust (T) along the rafter. The total pressure on the wall (R) is the resultant of $\frac{W}{2}$ and T acting at A; take AF to represent on scale 1875 lbs. and AH equal to CE; complete the parallelogram FH; then AK gives the magnitude and direction of the resultant R; it was found from fig. *h* that R equals 3885 lbs. and the angle KAF equals 16°; the results given by calculation are that R equals 3903 lbs., and that the angle KAF equals 16° 6′.

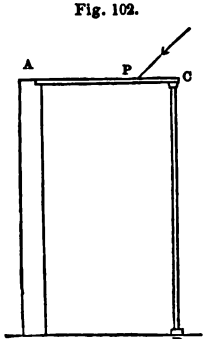
Ex. 378.—If in the last example the walls were 20 ft. high, $2\frac{1}{2}$ ft. thick, and of Portland stone, would they support the roof?

Ans. The wall will stand—the excess of the moment of the weight of 5 ft. of its length over that of the thrust being 17000.

Ex. 379.—If in the last example the walls be supported by buttresses 20 ft. apart from centre to centre, 15 ft. high, 2 ft. wide and $2\frac{1}{2}$ ft. thick, would these support the wall if its thickness were reduced to $1\frac{1}{2}$ ft.; and what would be the excess of the moment tending to support 20 ft. of the length of the wall over that which tends to overthrow it? *Ans.* (1) Yes. (2) 220000.

Ex. 380.—The pressure P of a roof is sustained as shown in the annexed diagram; AB, the outer wall, is of brick and is 20 ft. high and 18 in. thick, CD is an inner wall or column, on the top of which and of AB rests a horizontal rafter AC, which receives the pressure P; the distance AP is 7 ft. and AC is 10 ft.; the direction of P is inclined at an angle of 30° to the horizon. Determine the greatest value of P that AB can sustain, (1) when the friction on the top of the column is neglected, (2) when the coefficient of friction equals 0·5, the rafters being 10 ft. apart. *Ans.* (1) 1474 lbs. (2) 1853 lbs.

Fig. 102.



Ex. 381.—In the last example suppose the roof producing the pressure P to have a pitch of 30°, a span of 50 ft., and to weigh 20 lbs. per square foot, including the rafters; if the dimensions are the following will the wall stand? AB is of brickwork $2\frac{3}{4}$ ft. thick and 40 ft. high, the foot of the rafter P being 15 ft. distant from A; the top of the column being firmly fastened to C: the weight of AC and of the column are to be neglected, and the foot of the column supposed simply to rest on the ground.

Ans. It will stand—excess of moment per foot of length of wall being 7200.

Ex. 382.—In the last case suppose the distance of the column from centre to centre to be 20 ft. and also to be 20 ft. from the inside of the wall AB: determine the total pressure tending to crush one of the columns.

Ans. 8660 lbs.

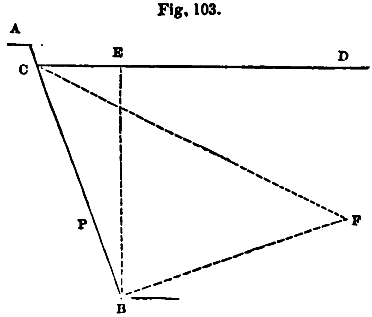
82. *Thrust on a tie-beam.*— Referring to Art. 81. it is evident that if the ends B and C of the rafters are tied together by a beam B C (which is called a tie-beam), the whole of the lateral pressure on the wall will be neutralised, since, whatever be the weight of the roof, the ends of the rafters cannot be forced outward so long as the joints are perfect: the roof will act merely by its weight, and there will be no tendency to overthrow, but only to crush the wall, except so far as the *bending* of the tie may give rise to a lateral pressure, a tendency which at present we do not consider, though in practice it is a very important element. The thrust on the tie-beam will of course be obtained by resolving the thrust along the rafter in a horizontal direction.

Ex. 383.—If two rafters A B and A C are each 20 ft. long, and their feet are tied by a wrought-iron rod B C whose length is 35 ft., a weight of 1 ton is suspended from A: determine the strain it produces on the tie, the weight of the rafters, &c. being neglected. If the rod have a section of a quarter of a square inch, determine the weight that must be suspended at A to break it. *Ans.* (1) 2024 lbs. (2) 18590 lbs.

Ex. 384.—There is a roof whose pitch is 22° 30′, the rafters are 40 ft. long; the weight of each square foot of roofing is 18 lbs.; determine the diameter of the wrought-iron tie necessary to hold the feet of the principal rafters with safety, supposing them 10 ft. apart. *Ans.* 1·28 inches.

83. *The pressure produced against a wall by water.*— The following construction can be very easily proved from the principles of hydrostatics. Let A B represent a section of the wall made by a vertical plane, C D the surface of the water; draw the vertical line B E; draw B F, a perpendicular to A B and equal to B E; join C F; then the pressure on any length of the wall will equal the weight of a prism of water whose base is C B F and height the

Fig. 103.



length of the wall; or, in other words, the pressure on each foot of the length of the wall will be the weight of as many cubic feet of water as the triangle B C F contains square feet; this pressure will act perpendicularly to the face of the wall through a point P, where $BP = \frac{1}{3} BC$.

Ex. 385.—There is a wall supporting the pressure of water against its vertical face; determine the pressure produced by the water on each foot of its length when 20 ft. of its height are covered. *Ans.* 12500 lbs.

Ex. 386.—In the last case determine the pressure on the lower 10 ft. of the wall. *Ans.* 9375 lbs.

Ex. 387.—An embankment of brickwork has a section whose form is a right-angled triangle ABC; the base BC is 6 ft. long; the height AB is 14 ft.; will the embankment be overthrown when the water reaches to the top, if AB is the face which receives the pressure?

Ans. Yes—the excess of the moment of pressure of water is 9767.

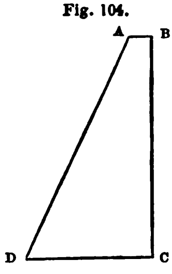
Ex. 388.—In the last case will the embankment be overthrown if AC is the face which receives the pressure?

Ans. Yes—excess of moment of weight of water 8675.

Ex. 389.—In Ex. 387 what horizontal pressure applied at A would keep the embankment steady? *Ans.* 698 lbs.

Ex. 390.—If the section of a river wall has the form shown in the accompanying diagram, in which AB = 5 ft., DC = 15 ft., and BC equals 50 ft.; BC being vertical, and the angles B and C right angles, find the height to which the water must rise against BC to overturn it. *Ans.* 37·2 ft.

Fig. 104.



Ex. 391.—If in the last example the dimensions were BC equal to 30 ft., AB equal to 3 ft., and DC equal to 10 ft., would the wall be overthrown if the water rose to the summit? *Ans.* Yes.

Ex. 392.—There is a cofferdam sustaining a pressure of 26 ft. of water, supported by props 20 ft. long, 20 ft. apart, the ends of which are placed at $\frac{2}{3}$rds below the surface of the water; determine the thrust on each. *Ans.* 468800 lbs.

Ex. 393.—If the section of an embankment of brickwork were of the form shown in Fig. 104., and the dimensions were AB equal to 4 ft., DC equal to 12 ft., and BC equal to 24 ft., would it support the water when it rises to the top and presses on the face AD?

Ans. Yes—excess of moment of weight of wall 5184.

Ex. 394.—If the coefficient of friction between the course of brickwork in the last example be 0·75, will the wall slide on its lowest section?

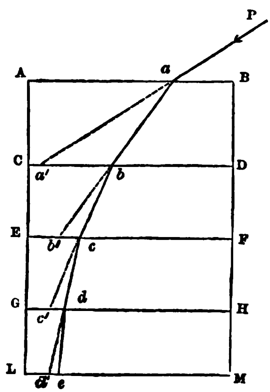
Ans. No—defect of horizontal pressure 2628 lbs.

Ex. 395.—In Ex. 392 what vertical pressure must by some means be supplied that equilibrium may be possible? *Ans.* 203100 lbs.

Section II.

84. *The line of resistance.*—Let A B L M represent any structure divided into horizontal courses by the lines C D, E F, G H .... and let it be subjected to the action of any pressure P along the line P a; produce P a to meet C D in a'; if the mass A B C D were without weight the pressure on C D would act on the point a'; but the total pressure on C D is the resultant (R_1) of P and the weight of A B C D; the direction of this resultant must cut C D at some determinate point between a' and D, say at b, and let the direction of R_1 be $b\,b'$; now the total pressure on E F will be the resultant (R_2) of R_1 and the weight of C D F E, which will cut E F at a determinate point c, between b' and F; in the same manner, the pressure on the point G H will act through a determinate point d, and on L M through a point e. Now if we join the points a, b, c, d ... we shall obtain a polygonal line which cuts each joint in the point through which passes the direction of the resultant pressure on that joint; if now we suppose the number of joints to be indefinitely great, the polygonal line will become a curved line, which is then called the line of resistance. It will be remarked that the direction of the resultants do not commonly coincide with the sides of the polygon $a\,b'$, $b\,c'$, ... and therefore the line of resistance determines only the point at which the pressure on each joint acts, not the direction of the pressure at that point.

Fig. 105.



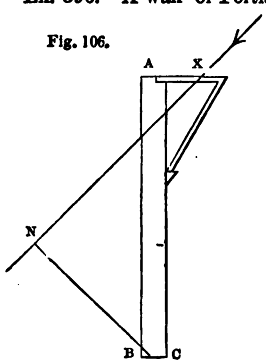
The line of resistance can be determined without much difficulty in a large number of cases: when this has been

done, the condition of equilibrium—so far as the tendency of the structure to turn round any of its joints is concerned—is that this line cuts each joint at a point within the structure; and, of course, the stability of a structure about any joint will be greater or less according as the intersection of the line of resistance with the joint be at a greater or less distance within the surface to which it is nearest.

It is plain that since the resultant of the pressures that act on a wall passes through the point of intersection of the line of resistance with its base, the algebraical sum of the moments of the pressures acting on the wall, taken with respect to that point, must equal zero. It may also be remarked that, in the case of most walls of ordinary shapes, the line of resistance continually approaches the extrados or outward surface; and hence, if the wall possesses a certain degree of stability with reference to its lowest joint, it will possess a greater degree of stability with reference to any higher joint; most of the following questions can, accordingly, be solved without the actual determination of the line of resistance.

Ex. 396.—A wall of Portland stone 30 ft. high and 2 ft. thick has to sustain on each foot of its length a thrust equal to the weight of 3 cubic feet of stone acting in a direction inclined to the vertical at an angle of 45°. Find the point of a bracket to which this pressure must be applied that the line of resistance may cut the base 6 in. within the extrados.

Fig. 106.



[Let the annexed figure represent a section of the wall; let the pressure act along the line XN, and let AX equal x; take BQ equal to 6 inches; then the condition of equilibrium is that the moments of the pressure and of the weight of the wall round Q be equal. Draw QN perpendicular to XN; it can be easily shown that

$$QN = AC \cos AXN - QC \sin AXN - AX \sin AXN$$

$$i.e.\ QN = \frac{28{\cdot}5 - x}{\sqrt{2}}$$

Whence we obtain

$$\frac{28 \cdot 5 - x}{\sqrt{2}} \times 3 = 60 \times \tfrac{1}{3}$$

$$\therefore\ x = 14 \cdot 36 \text{ft.}$$

It may be remarked that the determination of a perpendicular resembling QN occurs in many of the following questions. It may also be added that it is sometimes convenient to resolve the pressure into its horizontal and vertical components at X and obtain the moment of each.]

Ex. 397.—Determine the point of application of the pressure in the last article if the line of resistance cuts the base 3 in. within the extrados.

Ans. 7·04 ft.

Ex. 398.—A roof whose average weight is 20 lbs. per square foot, is 40 ft. in span and has a pitch of 30°; the walls of the building are of brickwork, and are 50 ft. high and 2 ft. thick; they are supported by triangular buttresses reaching to the top of the wall; the buttresses are 2 ft. wide, and 20 ft. apart from centre to centre. Determine their thickness at the bottom that the line of resistance may fall 6 in. within their extrados: determine also the answer that results from neglecting the weight of the buttress.

Ans. (1) 1·1675 ft. (2) 1·1754 ft.

Ex. 399.—A roof weighing 20 lbs. per square foot has a pitch of 60°; the distance between the walls that support it is 30 ft.; they are of Portland stone and are $2\frac{1}{2}$ ft. thick; the pressure of the roof being received on the inner edge of the summit, what is the extreme height to which the walls can be built? *Ans.* The wall can be carried to any height whatever.

Ex. 400.—If the weight of each square foot of a roof is 15 lbs., its pitch $22\frac{1}{2}°$, and the length of the rafters 30 ft., determine; (1) the thrust along the rafters, supposing them to be 4 ft. apart; (2) the strain upon the tie-beam if one is introduced; (3) the magnitude and direction of the pressure on each foot of the length of the wall-plate*, if there is no tie-beam; (4) the thickness of the wall, which is of brickwork and 20 ft. high, when the line of resistance cuts the base 2 in. within the extrados, the pressure of the roof being received on the inner edge of the summit; (5) determine the distance from the axis of the wall at which the pressure of the roof must act if the line of resistance cuts the base of the wall 3 in. within the extrados.

Ans. (1) 2352 lbs. (2) 2173 lbs. (3) 705 lbs. at an angle of 50° 21′ 20″ to the vertical. (4) 3 ft. (5) 2·7 ft.

Ex. 401.—If W is the weight supported by each rafter of an isosceles roof whose pitch is α; show that the thrust on each rafter is $\frac{W}{2 \sin \alpha}$, and the strain on the tie $\frac{W}{2 \tan \alpha}$

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* The wall-plate is the beam on which the feet of the rafters rest; its office is to distribute the pressure along the wall.

Ex. 402.—There is a river wall of Aberdeen granite 15 ft. high and having a rectangular section; the water comes to the distance of one foot from the top of the wall; find its thickness when the line of resistance cuts the base 6 in. within the extrados. *Ans.* 5·3 ft.

Ex. 403.—In the last example if the wall had a section of the form shown Ex. 390, where AB is 1 ft. long, the vertical face of the wall being towards the water; determine the width at the bottom when the line of resistance cuts the base 6 in. within the extrados. If the walls in this example and the last are 200 ft. long, determine the solid contents of each.

Ans. (1) 7·03 ft. (2) 15900 and 12045 cub. ft.

Ex. 404.—In each of the last examples determine the distance from the extrados of the point at which the line of resistance cuts a horizontal joint 8 ft. below the surface of the water. *Ans.* (1) 1·97 ft. (2) 1·74 ft.

[The point will, of course, be that round which the moment of the weight of the incumbent portion of the wall equals the moment of the pressure of the water on the eight feet.]

Ex. 405.—A river wall whose section is a right-angled triangle just supports the pressure of water when its surface is on a level with the top of the wall; show that the thickness of the base

$$= \text{height} \times \sqrt{\frac{w}{w_1 + 2w}}$$

if the hypothenuse of the triangle is turned towards the water; but when the perpendicular is turned towards the water the thickness of the wall

$$= \text{height} \times \sqrt{\frac{w}{2w_1}}$$

where w is the weight of a cubic foot of water, and w_1 that of a cubic foot of the material of the wall. And show from hence that the former is the more advantageous way of building such a wall when the specific gravity of the material of the wall is greater than 2, and the latter when it is less than 2.

Ex. 406.—A wall of brickwork is to be built round a reservoir 20 ft. deep; its slope is inward; it is one foot thick at top; what must be its thickness at the bottom, that, when the reservoir is full, the line of resistance may cut the base 6 in. within the extrados? *Ans.* 10·74 ft.

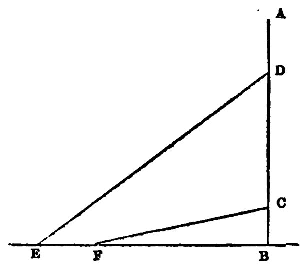
Ex. 407.—The wall of a reservoir full to the brim is of brickwork and is 20 ft. high and 2 ft. thick; it is supported by props at intervals of 6 ft.; the length of each is 20 ft., and its inclination to the horizon 30°: determine the thrust on each prop, its weight being neglected. *Ans.* 54632 lbs.

Ex. 408. In the last example determine the thickness of the wall that would just support the pressure of the water if the props were removed: if the wall stand on its lowest section without the aid of cement, what must be the coefficient of friction between the surfaces?

Ans. (1) 8·6 ft. (2) 0·65.

Ex. 409.—A cofferdam sustains the pressure of 26 ft. of water, and is supported at intervals of 10 ft. by props DE and CF; given that BC and BD are respectively 4 ft. and 18 ft. and that DE and CF are respectively 30 ft. and 18 ft.; find the thrust on each prop. And what must be the weight of the struts, and of the cofferdam, that the whole be not overthrown? The thickness of the cofferdam, and the adhesion at B, are to be neglected.

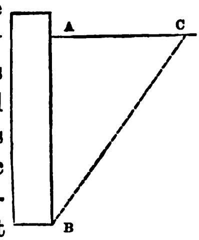
Fig. 107.



***Ans.* (1) Thrust on DE = 8020 lbs.; on EC = 144400 lbs. (2) 84900 lbs.**

85. *The pressure of earth.*—Let A B represent a section of a wall supporting earth, whose surface is A C, it is required to determine the pressure produced on A B by the earth. Now it must be remembered that two extreme cases may come under consideration: the first arises when the earth is thoroughly penetrated with water, in which case the pressure is the same as would result from hydrostatic pressure; the second arises when the cohesion of the earth is so considerable that it would stand with its face vertical even if the wall were removed. Dismissing these two extreme cases, let us suppose the wall A B removed, the following result will then ensue: the earth being friable will weather and break away until its surface has taken a slope B C, inclined to the horizon at an angle equal to the limiting angle of resistance; when reduced to this state it will have no further tendency to break away, and, unless washed down by rain, or removed by some other extrinsic cause, will remain permanently at rest at that slope, which is therefore called its *natural slope*. Hence, in the case we are considering, the wall is required to give a certain degree of support to the wedge of earth A B C; this wedge is generally sup-

Fig. 108.



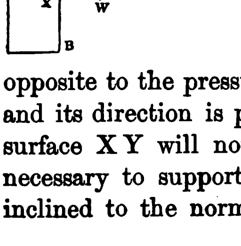
ported in some degree by the cohesion of its parts with each other and with the earth below B C, so that the wall will be sufficiently strong if it will support the earth, on the supposition that the cohesion is quite destroyed, unless (which is not contemplated) the earth should be saturated with water. The angle of the natural slope of fine dry sand is about 35°; of dry loose shingle about 40°; of common earth, pulverised and dry, about 45°.*

Proposition 19.

If w is the weight of a cubic foot of earth, and ϕ its natural slope, the pressure produced on the vertical face of a retaining wall by earth which does not rise above its summit, and which has a horizontal surface, is the same as that produced by a fluid the weight of a cubic foot of which is $w \tan^2\left(\frac{\pi}{4}-\frac{\phi}{2}\right)$.

Let A B be the section of the wall, B A C of the earth; take any portion A X equal to x of the wall, and suppose its length to be 1 foot; draw X Y, making an angle θ with the horizon greater than ϕ; then the weight W of the wedge A X Y equals $\frac{1}{2} w x^2 \cot \theta$, and acts vertically through a point P where $XP = \frac{1}{3} XY$, and is supported by the reaction R_1 of X Y and by the reaction R of the wall: the latter reaction is equal and opposite to the pressure produced by the earth on the wall, and its direction is perpendicular to A X; also, since the surface X Y will not exert a greater pressure than is just necessary to support A X Y, the direction of R_1 must be inclined to the normal to X Y at an angle equal to ϕ;

Fig. 109.



* See Mr. Moseley's Mechanical Principles of Engineering, p. 441.

also, the directions of R and R_1 must pass through the point P, in which W's direction cuts X Y, so that N X will equal $\frac{1}{3}$ of A X; moreover,

$$R : W :: \sin R_1 P W : \sin R_1 P R :: \sin(\theta - \phi) : \cos(\theta - \phi)$$
$$\therefore R = W \tan(\theta - \phi) = \tfrac{1}{2} w x^2 \operatorname{cotan} \theta \tan(\theta - \phi).$$

Now according as θ has different values R will have different values, and if we determine the value of θ for which R is greatest, the wall cannot be called on to supply a greater reaction, and this must therefore equal the pressure which A X actually sustains. But

$$\cot\theta \tan(\theta - \phi) = \frac{\cos\theta \sin(\theta - \phi)}{\sin\theta \cos(\theta - \phi)} = \frac{\sin(2\theta - \phi) - \sin\phi}{\sin(2\theta + \phi) + \sin\phi}$$
$$= \frac{1 - \dfrac{\sin\phi}{\sin(2\theta - \phi)}}{1 + \dfrac{\sin\phi}{\sin(2\theta - \phi)}}$$

Now since θ is essentially greater than ϕ, and each less than $90°$, the fraction $\dfrac{\sin\phi}{\sin(2\theta - \phi)}$ is, under all circumstances, less than unity, and will have its least value when $\sin(2\theta - \phi)$ has its greatest value; hence, when $\sin(2\theta - \phi)$ equals unity the above fraction is greatest. Since, then, its numerator is greatest and its denominator is least, *i. e.* the required value of θ is $\dfrac{\pi}{4} + \dfrac{\phi}{2}$, and, therefore, the required value of the pressure is

$$\tfrac{1}{2} w x^2 \operatorname{cotan}\left(\frac{\pi}{4} + \frac{\phi}{2}\right) \tan\left(\frac{\pi}{4} - \frac{\phi}{2}\right) = \tfrac{1}{2} w x^2 \tan^2\left(\frac{\pi}{4} - \frac{\phi}{2}\right)$$

acting through a point N which is below A by a distance

equal to $\frac{2}{3}x$; but this is the same as the pressure that would be produced by a fluid each cubic foot of which weighs $w \tan^2\left(\frac{\pi}{4} - \frac{\phi}{2}\right)$. Therefore, &c. Q. E. D.

Ex. 410.—A mass of earth the specific gravity of which is 1·7, whose surface is horizontal, presses against a revêtement wall whose top is on the level of the ground and height 20 ft., the natural slope of the earth being 45°; determine the pressure of the earth on each foot of the length of the wall.

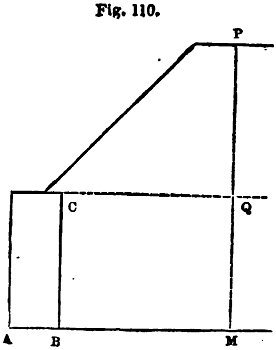
Ans. 3646 lbs.

Ex. 411.—If the wall in the last example is of brickwork and has a rectangular section, determine its thickness to enable it to sustain the pressure of the earth. *Ans.* 4·65 ft.

Ex. 412.—The vertical face of a revêtement wall of brickwork sustains the pressure of 20 ft. of earth the surface of which is horizontal and 2 ft. below the summit of the wall; the thickness of the wall at top is 1 ft.: what must be its thickness at bottom if it just sustains the earth; the specific gravity of the earth being 2 and its natural slope 45°? Also determine the thickness that would enable the wall to sustain the pressure if the earth were thoroughly permeated with water.* *Ans.* (1) 5·74 ft. (2) 10·1 ft.

Ex. 413.—If a pressure P is applied against a wall supported on the opposite side by earth with its surface horizontal; show that when P is on the point of causing the earth to yield, the resistance of the earth is the same as that of a fluid the weight of a cubic foot of which equals (weight of cubic foot of earth) $\times \tan^2\left(\frac{\pi}{4} + \frac{\phi}{2}\right)$.

Fig. 110.



* It is common for revêtement walls to sustain a surcharge of earth as shown in the accompanying diagram; an investigation of the pressure in this case will be found in Mr. Moseley's Mechanical Principles of Engineering, p. 453. The following practical formula (Morin, Aide-Mémoire, p. 417) gives the thickness (x) of a rectangular wall for a given height (H) of the revêtement (QM) and a surcharge (PQ) whose height is h, viz.

$$x = 0·865\,(H + h)\sqrt{\frac{w}{w_1}}\,.\tan\left(\frac{\pi}{4} - \frac{\phi}{2}\right)$$

w being the weight of a cubic foot of earth, and w_1 that of a cubic foot of masonry.

[The reasoning in this case is step by step the same as that given in Prop 19, except that now the wedge of earth is on the point of being forced *up*, so that the direction of R_1 will be on the other side of the perpendicular to XY.]

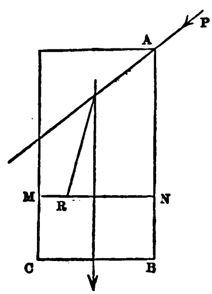
Ex. 414.—The wall of a reservoir of brickwork is 4ft. thick and 15ft. above the surface of the ground; the foundations are 15ft. deep; the natural slope of the earth is 45° and it weighs 100lbs. per cubic foot; when the reservoir is full (so that the water presses against the whole 30ft. of wall) will the wall stand, supposing the adhesion of the cement perfect?

Ans. Yes—excess of the moment of the greatest pressure that could sup port the wall over that of the pressure of the water 73480.

* Ex. 415.—If ABC is a section of a rectangular wall, P the pressure applied to every foot of its length at A, the inner edge of its summit; determine the equation to the line of resistance.

[Take any horizontal section of the wall MN; let $AN=x$, $BC=b$, then the weight W of $ANM=axw$, where w is the weight of a cubic foot of the wall; now if the direction of the resultant cuts MN in R, this will be a point in the line of resistance, and if $RN=y$ we are to determine a relation between x and y. The relation in question can easily be shown to be

Fig. 111.



$$awx\left(y-\frac{a}{2}\right)=P\,(x\sin\alpha-y\cos\alpha)$$

where α is the inclination of P's direction to the vertical.]

* Ex. 416.—In the last example show that the curve is a hyperbola and determine its asymptotes; and show that if the thickness of the wall equals $\sqrt{\dfrac{2P\sin\alpha}{w}}$ it may be carried to any height whatever with safety.

* Ex. 417.—If the wall in Ex. 415 has to support the pressure of earth or water reaching to the top of the wall; show that the line of resistance is a parabola with its axis horizontal, and its focus in the summit of the wall at a distance from the intrados equal to $\frac{a}{2}\left(1+\frac{3w}{w_1}\right)$, where w is the weight of a cubic foot of masonry and w_1 of the fluid.

* Ex. 418.—If ABCD is the section of a reservoir wall the vertical face of which (BC) is towards the water; the width of the top of the wall (AB) is a; the slope of AD is θ, and s is the specific gravity of the wall; show that when the water reaches to the top of the wall the equation to the line of resistance is

$$x^2\left(\frac{1}{s}+\tan^2\theta\right)-3\,xy\tan\theta+3\,ax\tan\theta-6ay+3\,a^2=0$$

* Ex. 419.—Show that if the wall in the last example stand, whatever be the depth of the water whose pressure it sustains, then $\tan\theta$ must be $> \frac{1}{\sqrt{2s}}$.

* Ex. 420.—Determine the equation to the line of resistance in a river wall of Aberdeen granite the thickness of which is 4 ft., and which sustains the pressure of water whose surface is on the level of the top of the wall. *Ans.* $x^2 = 63\,(y-2)$.

* Ex. 421.—Determine from the equation in the last example the height of the wall when the line of resistance intersects the base at a distance of 4 in. within the extrados. *Ans.* 10·2 ft.

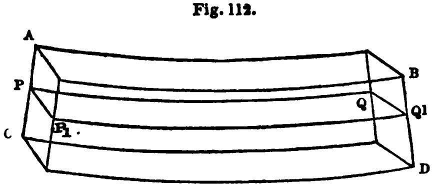
CHAP. VIII.

ON THE DEFLECTION AND RUPTURE OF BEAMS BY TRANSVERSE STRAIN.*

86. *Notation.*—The cases of deflection that we shall in the first place consider will be those of beams with a rectangular section; the following is the notation employed: a denotes the natural length of the beam, b its depth, and c its breadth, *i.e.* in a direction perpendicular to the plane of the paper; these measurements are supposed to be taken in inches, since the values of the modulus of elasticity E, given in Table III. p. 11, are referred to a square inch of section.†

87. *Neutral surface, and neutral line of a beam.*—If we consider a long beam of wood A D supported at its two ends, the effect of its weight will be to bend it into such a shape as is shown in the figure; it is evident that the under surface C D will suffer extension, and the upper surface A B com-

Fig. 112.



* This chapter cannot be read with advantage by any student who has not some acquaintance with the Integral Calculus.

† The term modulus of elasticity was introduced by Dr. Young, to whom is due the first correct investigations of the flexure of beams; his views are to be found in his Lectures, vol. ii. p. 46, &c.—He denotes the modulus by the letter m, which is equivalent to Ebc of the text: the reader will find the question fully discussed in Mr. Moseley's Mechanics of Engineering, Part V., which has been frequently referred to in drawing up the present chapter.

pression: so that there will be some section P Q which will be intermediate to the compressed and extended parts, having undergone neither compression nor extension; this surface is called the *neutral surface.* It sometimes happens that the whole of the substance is either compressed or extended; in such a case the neutral surface will not have a real existence, but there will exist without the body an imaginary surface bearing the same relation to the compressions or extensions as that borne by the actual neutral surface in other cases.

If we were to divide the beam into any number of thin parts by vertical planes parallel to P_1 B D, the forms of the surfaces would be unaffected, consequently any part of the neutral surface is like any other; we may therefore confine our attention to the section of that surface made by a vertical plane passing lengthwise through the centre of gravity of the beam: this section is called the *neutral line* of the beam; by the term *axis* of the beam is intended the geometrical axis of the beam considered as a prism. In the following examples it is assumed that the pressures act in a plane passing through the axis and parallel to the face of the beam. It is also assumed that the deflection of the beam is small, so that the moments of the pressures that bend it are not changed by the deflection of the beam.

Ex. 422.—If a line AB is subjected to a continuous pressure throughout its length of such a nature that the pressure at any point P is at the rate of k . AP per inch, then the resultant pressure equals $\frac{1}{2}k$. AB^2, and its moment round A equals $\frac{1}{3}k$. $\overline{AB}^3$.

[The solution is similar to that already given of the friction on a pivot.]

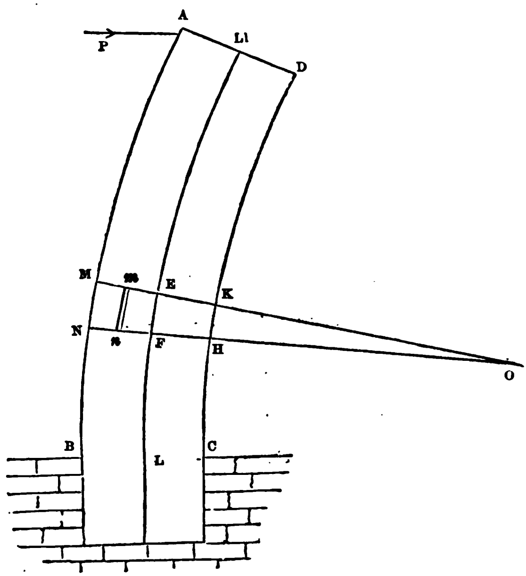
Proposition 20.

If a rectangular beam is held firmly by one end, and is acted on at the other by a pressure P, in a direction perpendicular to its length, the neutral line will coincide with the axis of the beam, and at any point distant p

inches from the end at which P acts the radius of curvature of that line equals $\frac{12\,P\,p}{E\,b^3\,c}$

Let A B C D represent the beam brought into its present position by the action of the pressure P; let L L_1 be the

Fig. 113.



neutral line; consider any small portion of the beam H K M N, which in its original state had the thickness E F, but owing to the action of P the ends M K and N H converge to O; we are to determine the position of the point F, and the distance F O; the former will give the position of the neutral line, the latter the radius of curvature at the point F.

We may suppose H M to be made up of thin laminæ parallel to E F, of which $m\,n$ represents one; all those within M F are in a state of extension, while those within

F K are in a state of compression. Now since each part of the beam is in equilibrium we may confine our attention to the portion M H, and may regard N H as a fixed surface; then the expansive pressures within F K, and the contractile pressures within F M must be in equilibrium with P. But it is plain that the contractile pressure of any lamina such as $n\,m$ acts in a direction perpendicular to that of P, and similarly of the expansive pressures of any lamina. Hence (Prop. 13.) the sum of the contractile pressures of M F = the sum of the expansive pressures of K F. Let E F and O F be denoted by l and ρ, N F and H F by b_1 and b_2 and nF by z, the width of the lamina being δz; now the natural length of $m\,n$ is l, therefore $m\,n - l$ is the extent by which it is stretched; therefore the pressure T necessary to produce this extension is given by the proportion (see Art. 6)

$$m\,n - l : l :: \frac{\mathrm{T}}{c\,\delta z} : \mathrm{E}$$

But by similar triangles $m\,n : l :: z + \rho : \rho$

$$\therefore m\,n - l : l :: z : \rho$$

$$\therefore \mathrm{T} = \frac{\mathrm{E}\,c}{\rho}\,z\,\delta z.$$

Now the pressure necessary to produce the extension equals that with which the lamina tends to contract, therefore T gives the contractile force of the lamina $m\,n$, and the same being true of all the others, their sum (by Ex. 422) will equal

$$\frac{\mathrm{E}\,c}{\rho} \cdot \frac{b_1^{\,2}}{2}.$$

and in like manner the sum of the expansive pressures will equal

$$\frac{\mathrm{E}\,c}{\rho} \cdot \frac{b_2^{\,2}}{2}.$$

And these being equal we have $b_1 = b_2$; also the same

will be true of any other section; the neutral line will pass along the middle of the beam, *i. e.* will coincide with its axis.

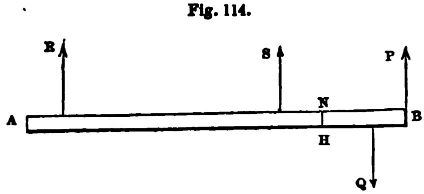
Next to determine ρ. It is evident that the expansive and contractile forces tend to turn A H in one direction round F, while the pressure P tends to turn it in the contrary direction round that point, and therefore the sum of their moments round that point will (Prop. 13.) equal the moment of P round the same point; but by Ex. 422 the former moments equal $\frac{E c}{\rho} \cdot \frac{b_1^3}{3}$ and $\frac{E c}{\rho} \cdot \frac{b_2^3}{3}$ respectively, and since $b_1 = b_2$ their sum will equal $\frac{E c}{\rho} \cdot \frac{b^3}{12}$; also the moment of P equals $P p$

$$\therefore \frac{1}{\rho} = \frac{12 P p}{E c b^3}$$

Cor. 1.—It will be remarked that in the above investigation no horizontal pressure has been introduced to balance P; in reality the horizontal pressure is supplied by the forces that hold the other end of the beam, *e. g.* the reaction of the brickwork if it is held as indicated in the figure.

Cor. 2.—If the beam were naturally horizontal and were kept at rest by any pressures, the results given in the above proposition are still true; thus if A B were the beam acted on by pressures P, Q, R, S, as shown in the figure; then if we take any

Fig. 114.

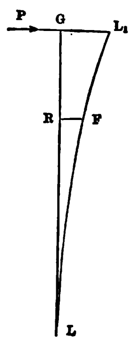


section N H, we may consider the part A N as held firmly by the forces, and the part B N bent, so that the radius of curvature corresponding to the section N H will equal $\frac{12 . (P p - Q q)}{E c b^3}$, $P p$ and $Q q$ being the moments of the

pressures P and Q round the middle point of N H. If we consider the part A N as bent, and B N as held firmly, we should obtain for the radius of curvature $\frac{12\ (\mathrm{R}\,r + \mathrm{S}\,s)}{\mathrm{E}\,c\,b^3}$ where R r and S s are the moments of R and S round the middle point of N H. It is evident that these two expressions give the same value for the radius of curvature, since $\mathrm{R}\,r + \mathrm{S}\,s = \mathrm{P}\,p - \mathrm{Q}\,q$ by Prop. 13.

Ex. 423.—Determine the equation to the neutral line of the beam considered in prop. 20.

Fig. 115.



[Let LL_1 be the neutral line, LG the position of the beam's axis when unbent, F any point in the neutral line, ρ the radius of curvature at F, x and y the coordinates of F, viz. LR and RF, we have

$$\frac{1}{\rho} = \frac{12\ \mathrm{P}\ (a-x)}{\mathrm{E}b^3c}$$

Now, since the curvature is small, $\frac{dy}{dx}$ is small, and therefore $\left(\frac{dy}{dx}\right)^2$ can be omitted; consequently

$$\frac{1}{\rho} = \frac{d^2y}{dx^2}$$

$$\therefore\ \frac{d^2y}{dx^2} = \frac{\mathrm{QP}}{\mathrm{E}b^3c}\ (a-x)$$

$$\text{whence}\quad y = \frac{12\ \mathrm{P}}{\mathrm{E}b^3c}\left(\frac{ax^2}{2} - \frac{x^3}{6}\right)\Big]$$

Ex. 424.—Show that the deflection of the beam in the last example equals $\frac{4\mathrm{P}}{\mathrm{E}bc} \cdot \frac{a^3}{b^2}$.

Ex. 425.—Show that the curvature of the neutral line increases gradually from L_1 to L; that in form it is "ultimately a cubic parabola, and that the depression is $\frac{2}{3}$rds of the versed sine of an equal arc in the least circle of curvature."*

[The equation obtained in Ex. 423 gives the ultimate form, since it is obtained on the supposition that $\frac{dy}{dx} = 0$].

Ex. 426.—If in Prop. 20 a pressure is applied to the end of the beam and

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* Young, vol. ii. p. 48.

gradually increased up to P, show that the number of units of work expended in producing deflection equals

$$\frac{2\,P^2}{E\,bc} \cdot \frac{a^3}{b^2}.$$

[Compare Ex. 149.]

Ex. 427.—The end of a beam of oak is firmly imbedded in masonry; the length of the projecting part is 15 ft., its breadth is 3 in. and its depth 6 in.; a pressure of 2 cwts. is applied perpendicularly at its end; determine the deflection, and the work expended in producing that deflection—the weight of the beam being neglected. *Ans.* (1) 5·5 in. (2) 51 units of work.

Ex. 428.—In the last example if the breadth of the beam were 6 in. and the depth 3 in., determine the deflection. *Ans.* 22·2 in.

Ex. 429.—If in Prop. 20 the beam in its natural state were horizontal and the bending pressure its own weight, show that $\frac{1}{\rho} = \frac{6\,w\,(a-x)^2}{E\,c\,b^3}$ where w is the weight of *one inch* of the length of the beam.

[The pressure producing the curvature at F is now the weight of AH.]

Ex. 430.—Show that the deflection in the last example is equal to $\frac{3}{2}\frac{wa}{E\,bc} \cdot \frac{a^3}{b^2}$.

Ex. 431.—Show that the deflection in the last example will be "half the versed sine of an equal arc in the circle of curvature at the fixed" end of the beam.*

Ex. 432.—If the beam in Ex. 430 were of elm, were 5 ft. long, 1 ft. broad, and 1 ft. deep, and had to support the pressure of brickwork 16 in. thick and 10 ft. high, determine the depression. *Ans.* 0·1708 in.

Ex. 433.—If a horizontal beam AB is supported at its ends and is loaded by a weight W at its middle point, and if ρ is the radius of curvature at a point on the neutral line whose distance from the middle point of the beam is x; show that

$$\frac{1}{\rho} = \frac{3\,W\,(a-2\,x)}{E\,c\,b^3}.$$

[The pressure producing the curvature is the reaction on the nearer point of support, *i. e.* a pressure $\frac{W}{2}$ acting at a distance $\frac{a}{2} - x$].

Ex. 434.—Show that the depression at the middle point of the beam in the last example equals $\frac{W}{4\,E\,bc} \cdot \frac{a^3}{b^2}$.

Ex. 435.—If in Ex. 433 the beam were bent by its own weight, and if w is the weight of one inch of its length, show that

$$\frac{1}{\rho} = \frac{3}{2} \cdot \frac{w\,(a^2 - 4\,x^2)}{E\,c\,b^3}.$$

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* Young, vol. ii. p. 49.

Ex. 436.—Show that the depression in the middle point of the beam is equal to $\frac{5}{32} \cdot \frac{wa}{Ebc} \cdot \frac{a^3}{b^2}$.

Ex. 437.—Show that "the depression in the middle of a bar supported at both ends, produced by its own weight, is five sixths of the versed sine of half the equal arc in the circle of least curvature." *

Ex. 438.—A fir batten 3 in. deep, $1\frac{1}{2}$ in. broad, is placed horizontally between two props 5 ft. apart and loaded with a weight of 135 lbs. in the middle; its own weight being neglected, determine the depression; determine also the depression if it were fixed at one end, loaded with the same weight at the other. *Ans.* (1) $\frac{18}{133}$ inches. (2) $\frac{288}{133}$ inches.

Ex. 439.—A spar of oak 3·2 in. square is placed horizontally between two props 12·8 ft. apart and loaded with 268 lbs. in the middle; determine the deflection, neglecting the weight of the beam. *Ans.* 1·597 in.

Ex. 440.—A piece of elm 2 in. square is placed horizontally between two supports 7 ft. apart, it is loaded in the middle with a weight of 125 lbs.; determine the deflection when its own weight is neglected. *Ans.* 1·65 in.

Ex. 441.—There is a beam of larch 6 in. deep, 4 in. wide, and 12 ft. long, it is supported on a fulcrum whose distance from one end is 4 ft.; the shorter end carries a weight of 2 cwts.; determine the deflection of each arm of the beam, its own weight being neglected. *Ans.* (1) 0·109 in. (2) 0·437 in.

Ex. 442.—The ends of a beam rest on horizontal supports, it is deflected by its own weight and a vertical pressure W acting through its middle point; determine the total deflection, and show that it equals the sum of the separate deflections produced by its own weight and by W, if W acts vertically downward, and their difference if W acts vertically upward.

Ex. 443.—If AB, AC are the principal rafters of a roof the feet of which are fastened together by a tie-beam BC, the middle point of which is D; if A and D are joined by a king-post which exactly neutralises the bending in the middle of the tie-beam caused by its weight, show that the strain on the king-post equals $\frac{5}{8}$ of the weight of the tie-beam.

Ex. 444.—In Ex. 439 determine the deflection when the weight of the spar is taken into account. *Ans.* 1·8 in.

Ex. 445.—A beam of larch supported at each end measures 20 ft. between the points of support, it is 6 in. wide and 10 in. deep, it sustains a wall of brickwork 30 ft. high and 1 ft. thick throughout its whole length—find the deflection. *Ans.* 23·13 in.

Ex. 446.—If the beam in the last example is supported by a column which exactly neutralises the deflection of the middle point; find the thrust on the column. *Ans.* 42170 lbs.

Ex. 447.—If in the last example the under surface of the beam in its un-

* Young, vol. ii. p. 49.

deflected state is 12 ft. from the ground, the middle point is supported by a column of cast-iron 3 inches in diameter, which in its uncompressed state is exactly 12 ft. long; determine the deflection of the beam and the thrust on the column. *Ans.* (1) 0·0504 in. (2) 42077 lbs.

[The column being compressible will allow the middle of the beam to descend, whereby the thrust on the column will be diminished: the question to be answered is at what degree of compression will the tendency of the column to recover its form upward exactly balance the tendency of the beam to deflect downward.]

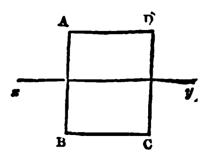
Ex. 448.—In the last example suppose the measurements to be made at 50° Fahrenheit, at what temperature would there be no deflection in the beam? *Ans.* 107° F.

88. *Deflection of beams whose sections are not rectangular.*—The reader will find little difficulty in extending the above investigation to the case of uniform beams whose sections have any form whatever; it can be proved that the neutral line passes through the centre of gravity of the section, and that the formula for the radius of curvature is

$$\frac{1}{\rho} = \frac{\Sigma\, P\, p}{E\, I}$$

Where $\Sigma\, P\, p$ denotes the sum of the moments of the pressures that tend to turn one portion of the beam round any section (round H N, for example, in the fig. to Prop. 20), E the modulus of elasticity, I the moment of inertia* about an axis passing through the centre of gravity of the section, and perpendicular to the plane in which the forces act. In fact the formula obtained in Prop. 20 is only a particular case of the above formula, since, in the case of a rectangle A B C D in which A B $= b$ and B C $= c$, the moment of inertia about an axis $x\, y$ perpendicular to A B, and passing through the centre of gravity of the rectangle, equals $\dfrac{b^3 c}{12}$.

Fig. 116.



The reader will find the case of the flexure of beam

* For the definition of the moment of inertia, see Part II.

developed from the above fundamental formula in Mr. Moseley's Mechanical Principles of Engineering. The following examples are all that our limits permit.

Ex. 449.—If a hollow cylinder the radii of whose section are r_1 and r be supported horizontally at two points whose distance is a; show that when it sustains a weight W at its middle point, the radius of curvature of the neutral line at a point distant x from the middle is given by the formula

$$\frac{1}{\rho}=\frac{W\ (a-2x)}{\pi E\ (r_1^4-r^4)}$$

and the deflection at the middle point by the formula

$$\delta=\frac{W\ a^3}{12\pi E\ (r_1^4-r^4)}.$$

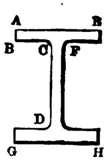
Ex. 450.—If in the last example the cylinder sustains throughout its length a uniform pressure of w lbs. per inch, then

$$\frac{1}{\rho}=\frac{w\ (a^2-4x^2)}{2\pi E\ (r_1^4-r^4)}$$

$$\text{and}\quad \delta=\frac{5\ wa^4}{96\pi E\ (r_1^4-r^4)}.$$

Ex. 451.—If an iron girder* has a section of the form shown in the annexed diagram, of the following dimensions, $AE=c_1$, $AB=b_1$, $CF=c$, $CD=b$, the lower end GH being of the same dimensions as the upper, show that when this girder sustains a uniform pressure throughout the whole of its length the deflection at the middle point is given by the formula

Fig. 117.



$$\delta=\frac{5\ wa^4}{32\left\{6\ (b+b_1)^2\ b_1\ c_1+2b_1^3\ c_1+b^3c\right\}E}$$

Ex. 452.—If there are two beams containing the same amount of materials, of the same length and the same depth, and sustaining the same weight, the one has a rectangular section, the other a section of the form shown in the last example; given that $b=4$ in., $c=1$ in., $b_1=1$ in., $c_1=4$ in., show that the deflection of the rectangular beam will be $\frac{14}{9}$ths of the deflection of the other beam.

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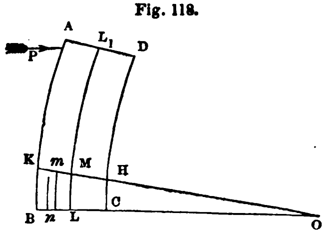
* In practice the lower flange is commonly made much larger than the upper, since cast-iron is much stronger in resisting a crushing pressure than a strain, and of course the greatest economy of materials is effected when the pressure that would tear the lower flange would also crush the upper: to discuss this question would lead us beyond our present limits: see Mr. Moseley's Mechanical Principles, p. 556, Mr. Rankine's Applied Mechanics, p. 319. See also Mr. Fairbairn's Useful Information, Append. I.

89. *Rupture of a rectangular beam.*—Referring to Prop. 20, it has been remarked that the curvature of the beam becomes progressively greater in going from L_1 to L, consequently the extension of the substance is greatest at B, and when rupture occurs it must result from the extension of the substance at B being greater than it can bear. Let us suppose that a pressure of S lbs. per square inch will produce just that degree of extension at which rupture ensues, and let us examine the state of a small portion of the beam at B C, the natural length of which is l; construct a figure similar to that in Prop. 20, and use the same notation; suppose B L to be divided into a number of parts each equal to δz; now, as the lamina at B K is on the point of breaking, it must be stretched by a pressure of S lbs. per square inch, and if its extension is δl we shall have $\delta l = \dfrac{S l}{E c \delta z}$

Fig. 118.



and if we consider the extension $\delta' l$ of any other lamina $m n$, whose distance L n from L equals z, we shall have

$$\delta' l : \delta l :: z : \frac{b}{2}$$

Now the contractile pressure of this lamina (Q) is given by the equation

$$\delta' l = \frac{Q l}{E c \delta z}$$

$$\therefore Q = \frac{2 z}{b} S$$

and the expansive pressure of any lamina between L and C will be given by the same formula. Now the moment of P round L must equal the sum of the moments of the

contractile pressures of the laminæ between B and L and those of the expansive pressures of the laminæ between L and C; these moments are respectively P a, $\frac{1}{12}$ S $c\,b^2$ and $\frac{1}{12}$ S $c\,b^2$ (by Ex. 422), and therefore the pressure P producing rupture is given by the equation

$$P = \frac{S}{6} \cdot \frac{cb^2}{a}.$$

The coefficient S is termed the modulus of rupture; it is not the same as the tenacity of the substance, but is closely related to it. The following table* gives the value of S for certain substances: —

TABLE XIV.

MODULUS OF RUPTURE.

Substance.	lbs. per square inch.	Substance.	lbs. per square inch.
Oak (English)	10032	Fir (Riga)	6612
Larch	4992	Elm	6078

Ex. 453.—If a horizontal beam, whose weight is neglected, is supported at its extremities and subjected to the action of a vertical pressure P at its middle point, it will break (across its middle section) when

$$P = \frac{2\,S}{3} \cdot \frac{cb^2}{a}.$$

Ex. 454.—If a horizontal beam is supported at one end, and every inch of its length sustains a pressure w, show that the beam is on the point of breaking when

$$w = \frac{S}{3} \cdot \frac{cb^2}{a^2}.$$

Ex. 455.—If in the last example the beam had been supported at both ends, show that

$$w = \frac{4S}{3} \cdot \frac{cb}{a^2}.$$

Ex. 456.—What load applied at the centre of a beam of oak, 20 ft. long,

* From Mr. Moseley's Mechanical Principles of Engineering, p. 622. For further information on the subject of the text the reader is referred to that work and to Mr. Rankine's Applied Mechanics.

3 in. deep, and 4 in. wide will be sufficient to produce rupture?—Its own weight being neglected. *Ans.* 1003 lbs.

Ex. 457.—In the last example to what height could a wall of brickwork 1 ft. thick and resting on the oak be carried before producing rupture?—And to what if the depth of the beam were increased to 9 in.?

Ans. (1) 10·7 in. (2) 96·7 in.

Ex. 458.—A beam of larch 1 ft. square has its end firmly imbedded n masonry from which it projects 7 ft.; to what height could a wall of brickwork 2 ft. thick resting on this beam be carried without producing rupture?

Ans. 21·8 ft.

Ex. 459.—A beam whose weight is W, when supported at the ends in a horizontal position, will just break under a pressure P applied at its middle point, show that

$$P=\frac{2S}{3}\cdot\frac{cb^2}{a}-\frac{W}{2}.$$

Ex. 460.—If a beam AB whose length is a is supported at its ends in a horizontal position and sustains a pressure of P lbs. at a point C such that $AC=a_1$ and $BC=a_2$, and if X is any section at a distance x from B, show that the moment tending to produce rupture round X equals $\frac{P\,x a_1}{a}$ when X is between B and C, and equals $\frac{P\,(a-x)\,a_2}{a}$ when X is between A and C; show also that the moment tending to produce rupture round C equals $\frac{P\,a_1\,a_2}{a}$.

Ex. 461.—Show that in the last example the pressure which acting at C will produce rupture is given by the formula

$$P=\tfrac{1}{6}S.\frac{acb^2}{a_1\,a_2}$$

and that the smallest pressure that can produce rupture must act at the middle point of the beam.

* Ex. 462.—Given a cylindrical log of wood, show that the strongest rectangular beam that can be cut out of it is one whose sides are in the proportion of $1 : \sqrt{2}$.

Ex. 463.—A beam of oak is supported in a horizontal position on points 20 ft. apart, it is 3 in. deep and 4 in. wide; determine the weight that can be suspended at a distance of $6\frac{2}{3}$ ft. from one point of support without breaking it? What would be the magnitude of the weight if the depth were 4 in. and breadth 3 in.? *Ans.* (1) 1128·6 lbs. (2) 1504·8 lbs.

Ex. 464.—What must be the depth of a beam of Riga fir 4 in. wide, 30 ft. long, that will just sustain a weight of $\frac{1}{2}$ a ton at its middle, taking into account its own weight? *Ans.* 5 in.

CHAP. IX.

OF MACHINES IN A STATE OF UNIFORM MOTION,—OF TOOTHED WHEELS,—AND OF VIRTUAL VELOCITIES.

90. *Fundamental Principles.* — The questions connected with the important case of machines in a state of uniform motion occupy a position intermediate to those of statics and dynamics; this position is best defined with reference to the work done by the pressures that act on the machine. On the one hand, it is evident that the pressures must satisfy the conditions of equilibrium, for otherwise they could be replaced by their resultant*, and an acceleration or retardation of the motion would ensue; on the other hand, the case differs essentially from a case of equilibrium in the circumstance that each particular pressure does work, since in general the point of application of each pressure will move, and thus arises that combination of pressure and motion which is intended by the term "work." Again, since the motion does not undergo either acceleration or retardation, the work done by those pressures which tend to accelerate the motion must equal that done by those pressures which tend to retard the motion; for, if one were larger than the other, the work accumulated in the machine would either be increased or diminished, and an acceleration or retardation of its motion would ensue. In the present chapter we shall, therefore, assume the following fundamental principles: —

(1.) If a machine is in a state of uniform motion, the pressures which act on it satisfy the conditions of equilibrium.

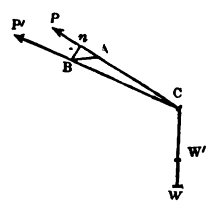
* Or generally their two resultants.

(2.) The algebraical sum of the works done by the several pressures will equal zero.

91. *Definition of the work done by a pressure.*— We have hitherto supposed (see Chap. II.) that the point of application of the pressure which does work moves in the straight line along which the pressure acts. It will be found that the definition becomes perfectly general when stated thus:—The work done by a constant pressure is the product of that pressure and the space described by its point of application *measured in the direction of the pressure.* The object of the present article is to show that this is the correct generalisation of the original definition, and to fix the meaning of the words "measured in the direction of the pressure."

(*a.*) Suppose a weight W to be attached to the end of a perfectly flexible and inextensible string without weight, and suppose it to pass over a smooth point C; let W be balanced by a pressure P acting at A along AC, then will P equal W; now, suppose the point A to be moved through a small space from A to B, draw B n at right angles to AP, then W is caused to ascend to W′, and An is ultimately equal to W W′; now the work expended in raising W equals W × W W′, *i.e.* it ultimately equals P × A n, which is therefore ultimately the work done by P. Hence, if the point of application of a pressure P is moved through any small space, and that space is projected on the original direction of the pressure into a line whose length is p, then the work done by the pressure will ultimately equal P p. It will be remarked that the line representing the pressure (Art. 27) would be measured from A towards P; consequently n falls upon the line which represents the

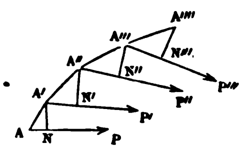
Fig. 119.



pressure, and in this case it is usual to consider the work done by P as positive, and as negative if n had fallen on the other side of A. The reason of this is evident from the fact that, in the former case, W is caused to ascend, in the latter it is caused to descend.

(*b*.) Let it be supposed that the point of application of the pressure P is transferred successively to $A, A', A'', A''' \ldots$ the successive directions of the pressure being AP, $A'P'$, $A''P''$, ... let fall on them the perpendiculars $A'N$, $A''N'$, $A'''N''$ .... and let the lines AN, $A'N'$, $A''N''$ .... be denoted by p, p', p'', ... then the work done by P will equal the limit of the product

Fig. 120.



$$P \times (p + p' + p'' + \ldots).$$

and the ultimate value of $p + p' + p'' + \ldots$ is the space described by P's point of application measured in the direction of the pressure. As this definition is very general, it will be well to particularise the more common cases.

(1.) If the point of application of the pressure moves in any line, straight or curved, while the successive positions of the pressure are all parallel, then the work done will equal the product of the pressure, and the projection of the line on the direction of the pressure.

(2.) If the direction of the pressure is always a tangent to the curve described by its point of application, the work done equals the product of the pressure and the length of the curve.

(*c*.) It sometimes happens that the pressure itself varies while its direction makes a variable angle with the curve described by its point of application: an example of this case is supplied by the motion of a crank. It will be evident on considering paragraph (*b*) that the work done in *this case will* equal the limit of the sum of $Pp + P'p' +$

$P''p'' + \ldots\ldots$ where P, P', P'' are the successive magnitudes of the pressure.*

92. *Cases in which pressures do no work.*—The chief of these are the following:—

(*a.*) When the point of application of a pressure does not move, *e. g.* when a body revolves round an axis that can be considered as a straight line, the reaction of that axis does no work; again, when a body rolls without any sliding on a surface, the reaction does no work.

(*b.*) When the point of application of the pressure moves at each instant in a direction perpendicular to that of the pressure: *e. g.* the reaction of a smooth surface does no work in the case of sliding motion.

(*c.*) If amongst the various pressures concerned there exists any subordinate system which is separately in equilibrium, it will, of course, do no work: the internal pressure of a rigid body constitute such a system; if however, when a system of pressures is applied to a body, it is compressed or extended, the work done by the internal pressures cannot be neglected (Ex. 149).

93. *The modulus of a machine.*†—In most simple machines there is a certain pressure or *weight* Q, and a second pressure or *power* P, whose function is to overcome the former, the machine itself being the means by which the one pressure is enabled to act on the other: in consequence of the intervention of the machine there will, in general, be a number of passive resistances called into play, but still the relation between P and Q can generally be expressed by means of an equation of the form

$$P = AQ + B \qquad (1)$$

* If in this case P' is the resolved part of the pressure along the curve, and ds the length of an element of the curve, then the expression for the work done will be $\int P' \, ds$.

† The reader who wishes for more information on this subject is referred to Mr. Moseley's Mechanical Principles of Engineering, Part III.

where A and B denote quantities depending on the form of the machine and on the passive resistances: the particular values of A and B in the cases of the inclined plane, screw, pulley, &c., have been already given. (Ex. 269, 288, 353, &c.)

Now suppose that while P's point of application describes a space s_1, Q's describes a space s_2, these spaces being measured in the directions of the pressures, then Ps_1 is the work done by P and Qs_2 the work done by Q: let the former of these be denoted by U_1 and the latter by U_2; the algebraical equation which expresses the relation between U_1 and U_2 is the modulus of the machine. The following is a general method of determining the modulus. From equation (1) obtain the relation that would subsist between P and Q if all the passive resistances were neglected, and let the relation be

$$P' = A_0 Q \qquad (2)$$

where Q has the same value in both equations, and consequently P and P′ have different values. Now if there were no passive resistances, the work done by P′ would equal the work done by Q, and consequently

$$P's_1 = Qs_2 \qquad (3)$$

$$\therefore \quad s_2 = A_0 s_1 \qquad (4)$$

But if we multiply equation (1) by s_1, we shall obtain

$$Ps_1 = AQs_1 + Bs_1$$

and ∴ from equation (4)

$$Ps_1 = \frac{A}{A_0} Qs_2 + Bs_1$$

$$\text{or } Ps_1 = \frac{A}{A_1} . Qs_2 + \frac{B}{A_0} . s_2 .$$

$$\therefore \quad U_1 = \frac{A}{A_0} U_2 + Bs_1 \qquad (5)$$

$$\text{or } U_1 = \frac{A}{A_0} U_2 + \frac{B}{A_0} . s_2 \qquad (6)$$

The equation (5) will be employed when s_1 is given, and the equation (6) when s_2 is given.

Ex. 465. In the inclined plane, determine the relation between the works done by P and Q when the mass Q is moved over a certain space.

The relation between P and Q when the former is on the point of preponderance is

$$P = Q\frac{\sin(\alpha + \phi)}{\cos.(\beta - \phi)}$$

Therefore P′, the pressure required to balance Q if there were no passive resistances, is given by the equation

$$P' = Q\frac{\sin\alpha}{\sin\beta}$$

Now if p is the space through which P's point of application moves measured in the direction of that pressure; and if similarly q is Q's space, we have

$$P'p = Qq$$

$$\therefore q\sin\beta = p\sin\alpha^*$$

$$\text{But } Pp = Qp\frac{\sin(\alpha + \phi)}{\cos(\alpha - \phi)} = Qq\frac{\cos\beta\sin(\alpha + \phi)}{\sin\alpha\cos(\beta - \phi)}$$

$$\therefore U = U_1\frac{\cos\beta\sin(\alpha + \phi)}{\sin\alpha\cos(\beta - \phi)}$$

Ex. 466.—If the coefficient of friction between certain bodies and an inclined plane be 0·10, the angle of the incline is 12°, and the pressure acts at an angle of 5° with the plane, determine the number of units of work that must be done, for every million of useful units that are yielded.

Ans. 1457000.

Ex. 467.—If in the last example the weight is 1000 lbs., over what length of the plane must it be drawn before 1000000 units of work are done by the agent? *Ans.* 3299 ft.

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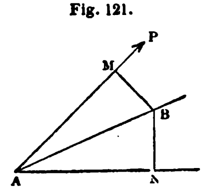
* It is evident that this must be the relation between p and q; for let AB be the space over which the body has been moved by a pressure acting parallel to AP, draw BM perpendicular to AP and BN perpendicular to AN, then p=AM and q=BN, and it is plain that

$$p = AB\cos BAM$$
$$q = AB\sin BAN.$$

Fig. 121.



Ex. 468.—Show that the number of units of work that must be expended in drawing the weight Q over a length l of the plane by means of a pressure parallel to the plane equals

$$Q\,l\,\frac{\sin(\alpha+\phi)}{\cos\phi}$$

and that this equals the number that must be expended in dragging it along the *base* of the plane (supposing it equally rough) and then lifting it vertically through the height of the plane.*

Ex. 469.—If the coefficient of friction is μ and the incline 1 in m show that, when l is the *horizontal length of the plane*, the number of units of work equals

$$Q\,l\,(\mu \pm \frac{1}{m})$$

The positive sign being used when the incline is upward, the negative when it is downward.

Ex. 470.—If a train weighs 80 tons and the friction is 7 lbs. per ton, determine the number of units of work that must be expended in drawing it for 4 miles over an incline of 1 in 200; and determine the horse-power of the engine that will do this in 10 minutes with a uniform velocity.

Ans. (1) 30750720 U. W. (2) $93\frac{2}{11}$ H.-P.

Ex. 471.—In the last example over what space on a horizontal plane would the same engine have drawn the train in the same time?

Ans. $10\frac{2}{3}$ miles.

Ex. 472.—How long would it take the engine in Ex. 470 to draw the same train with a uniform velocity over a space of 4 miles on an incline of 1 in 100? *Ans.* $16\frac{2}{13}$ min.

Ex. 473.—A train is drawn with a uniform velocity over an incline 3 miles long of 1 in 250, on which the resistances are 7 lbs. per ton, determine the distance on a horizontal plane which the same train could be drawn over with a uniform velocity by the same expenditure of force.

Ans. $6\frac{21}{25}$ miles.

Ex. 474.—If the resistances are 9 lbs. per ton, determine the weight of the train which an engine of 100 horse-power could draw up an incline of 1 in 540 with a velocity of 30 miles an hour. *Ans.* 95 tons.

Ex. 475.—If a pivot sustaining a pressure of Q lbs. is made to revolve

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* It may be remarked that the number of units of work done by P, if a machine is in a state of uniform motion, equals the work that must be expended upon Q under any circumstances; if P does fewer than the required number of units, that number is made up from the work previously accumulated in the machine whose motion will be retarded; if it does more, the extra units will be accumulated and the motion will be accelerated.

once, show that the number of units of work expended on the friction of the end equals $\frac{4}{3}\pi\mu\rho Q$.

Ex. 476.—In the case of a single fixed pulley the number of units of work expended in raising a weight Q through q feet is given by the formula

$$U = a\,Qq + bq$$

where a and b have the values assigned in Art. 74.

Ex. 477.—In the case of a tackle of n sheaves show that the number of units of work expended in raising a weight of Q lbs. through q feet is given by the formula

$$U = Qq.\frac{n\,a^n\,(a-1)}{a^n-1} + \left(\frac{nb\,a^n}{a^n-1} - \frac{b}{a-1}\right) nq.$$

[See Ex. 355.]

Ex. 478.—In Ex. 357 determine the number of units of work expended on the passive and on the useful resistances when the weight of 1000 lbs. is raised through 50 ft. *Ans.* (1) 90700. (2) 50000.

Ex. 479.—"It is said that in a pair of blocks with five pulleys in each two thirds of the force are lost by the friction and rigidity of the cords." * Determine the degree of truth in this statement when each sheave is 4 in. in radius, and turns of an axle $\frac{1}{2}$ of an inch in radius, the axle being of wrought iron and the bearing of cast iron, and the rope 4 in. in circumference; the weight to be raised being 1000 lbs.

$$Ans.\ \frac{\text{Work expended on passive resistances}}{\text{Work done}} = \frac{19}{29} \text{ nearly.}$$

Ex. 480.—In the capstan Ex. 363 show that the work that must be done by the pressures in order to move the weight Q through a space q is given by the formula

$$U = \left(1 + \frac{r\sin\phi}{b}\right)\left(1 + \frac{B}{b}\right) Qq + \frac{qA}{b}\left(1 + \frac{r\sin\phi}{b}\right) + \frac{2\mu_1 rW}{3} \cdot \frac{q}{b}.$$

Ex. 481.—A rope passes over a single fixed pulley in such a manner that its two parts are at right angles; the one end carries a weight Q; the radius of the pulley is r and of the axle ρ, the angle β such that $\sin\beta = \frac{\rho\sin\phi}{r\sqrt{2}}$ then, the weight of the pulley being neglected, show that if P is the pressure that will just raise Q, we have

$$P = \left(Q + \frac{A + BQ}{r}\right) \tan(45^\circ + \beta).$$

Ex. 482.—In the last example show that the relation between P and Q may be very nearly represented by the formula

$$P = Q\left(1 + \frac{B}{r} + \frac{\rho\sqrt{2}}{r}\sin\phi\right) + \frac{A}{r}\left(1 + \frac{\rho\sqrt{2}}{r}\sin\phi\right).$$

* Dr. Young's Lectures, p. 206.

Ex. 483.—A weight of 500 lbs. has to be raised from a depth of 50 fathoms; it is fastened to a rope which passes over a fixed pulley in such a manner that the parts of the rope are at right angles to each other; the rope is wound up by means of a capstan which is turned by two equal parallel pressures acting at the end of equal arms; the rope is 3 in. in circumference, the pulley 6 in. in effective radius, its axle half an inch in radius, and of wrought iron turning upon cast; the capstan weighs 4 cwts., its axle is 4 in. in radius, oak moving on wrought iron, the effective radius of the capstan 15 in.; determine the number of units of work that must be done in order to raise the weight, and the number expended on passive resistances.

Ans. (1) 204356. (2) 54356.

Ex. 484.—If a rope *rests* upon a fixed pulley and is pulled by two horizontal pressures P and Q, and if W is the weight of the pulley and of the rope supported by it, show that when P is the preponderating pressure

$$P = Q\left(1 + \frac{B}{r}\right) + \frac{A + W\rho \sin\phi}{r}.$$

Ex. 485.—If the equation in the last article be written $P = KQ + L$ and if the rope passes over n equal pulleys whose centre are in the same line, show that

$$P = Q \,.\, K^n + \frac{L(K^n - 1)}{K - 1}.$$

Ex. 486.—A winding engine is employed to raise weights from a mine shaft; the rope passes at right angles over a fixed pulley, and then horizontally over n fixed pulleys; determine the number of units of work that must be expended in raising a weight Q through q feet by means of them.

[The answer is readily obtained by means of Ex. 481 and 485.]

Ex. 487.—The rope of a winding engine passes over 20 pulleys before it reaches the shaft; it is drawn tight and presses on them only by its weight; at the shaft it passes at right angles over another equal pulley; they are 15 ft. apart; each weighs 56 lbs. and is 2 ft. in effective diameter, its axle is 1 in. in diameter, wrought iron turning on cast; the rope is 4 in. in circumference; how many units of work must be expended in raising a weight of 1000 lbs. through 150 fathoms, the weight of the first pulley being neglected?

Ans. 3,313,000 units of work.

[If we represent the weight supported by Q, then the pressure that will bring this into the state bordering on motion when the rope has passed at right angles over the first pulley being represented by Q′ we have from Ex. 481

$$Q' = 1{\cdot}058\,Q + 5{\cdot}5611.$$

and if P be the pressure that will bring Q′ into the state bordering on motion after the rope has passed over the 20 pulleys, we have from Ex. 485,

$$P = 2{\cdot}482\,Q' + 191.$$
$$\therefore\ P = 2{\cdot}626\,Q + 205.$$

Now if x is the depth in feet of the weight below the surface at any instant

$$Q = 1000 + 0{\cdot}72x.$$

$$\text{or} \quad P = 2831 + 1{\cdot}89x.$$

And then the units can be found by Prop. 1.]

Ex. 488.—There is a fixed pulley 20 inches in radius moving on an axle 2 inches in diameter ($\sin\phi = 0{\cdot}15$); a weight of 500 lbs. is raised from a depth of 300 feet by means of a rope 3 inches in circumference which passes over it; the end of the rope falls as the weight rises; determine the error that results from neglecting the weight of the rope in calculating the units of work required to raise the weight—the united length of the two hanging parts of the rope being reckoned at 300 ft. *Ans. Error* $= 155507 - 154962 = 545$.

[Compare Ex. 158.]

Ex. 489.—In the last example determine the error that would result from neglecting the weight of the rope if the end were *not* allowed to fall.

Ans. Error $= 173760 - 154962 = 18798$.

Ex. 490.—If a weight Q is raised through a height q by means of a screw, show that if the same notation is employed as in Ex. 297 the units of work expended is given by the formula

$$U = Qq\left\{\tan(\alpha+\phi) + \tfrac{2}{3}\cdot\frac{\rho}{r}\,\mu\right\}\text{cotan}\,\alpha$$

where all frictions are neglected except those between thread and groove and on the end of the screw.

Ex. 491.—An iron screw 4 in. in diameter communicates motion to an iron nut, the screw thread is inclined to its base at an angle of 18°, the diameter of the end of the screw 2 in.; all the surfaces are of cast iron; determine the number of units of work that must be expended in raising a weight of 3 tons through a height of 2 ft. by means of this screw.

Ans. 23358.

Ex. 492.—Determine through what height a man working with this screw could raise a weight of 1 ton in a day; and what would be the best length of the arm of the screw on which he works—pushing horizontally; determine also the part of his work which is expended in overcoming friction.

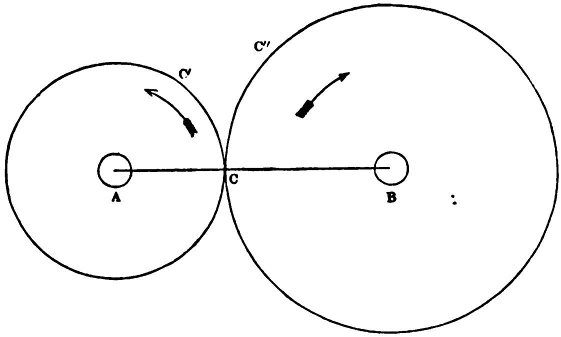
Ans. (1) 388·4 ft. (2) $7\frac{3}{4}$ ft. (3) rather more than $\frac{3}{7}$th.

Toothed Wheels.

94. *The end to be attained by cutting teeth on wheels.*—The problem to be solved is this: given an axle A, moving with a uniform angular motion round its geometrical axis, it is required to connect it in such a manner with a parallel axle B, as to communicate to it a uniform angular motion which shall have a given ratio to the

former. This problem is solved as follows: suppose the axle A to revolve m times in one minute, and it is required to make the axle B revolve n times in one minute;

Fig. 122.



join the centres A and B, divide AB into $m+n$ equal parts, and take AC equal to n of these parts, and therefore BC will contain m of them, so that

$$AC : CB :: m : n$$

with centres A and B, and radii AC, BC respectively, describe circles touching at C; if these circles are fixed each to its own axle, and revolve with them, and if their circumferences are rough, so that they roll on each other, the problem is solved; for take on the circumferences respectively points C′ and C″ which were in contact at C, then must the arc CC′ equal the arc CC″, since the several points of the arcs have been successively in contact each with each, and this is true whatever be the lengths of those arcs. Now, in one minute the point C′ describes an arc whose length is 2π AC . m, and therefore C″ describes an arc whose length is 2πAC . m, and therefore an arc whose length is 2π BC . n, since AC . m=BC . n; but 2πBC . n is n times the circumference of the circle whose radius is

BC, and therefore the axle B moves in the required manner.

It is evident that the angular motions will have the same ratio whatever be the duration of the time; and hence when the time is very short, so that if the angular motion of the axle A varies from instant to instant, that of the axle B will also vary, but will maintain the same constant relation to that of the angular motion of A.

It is also plain that the directions of the angular motions will be contrary, as indicated by the arrow heads.

It may be remarked that the wheel AC is called the driver, and BC the follower.

Ex. 493.—If in the last article a single wheel moving on a parallel axle with its centre in the line AB were interposed between AC and BC, it would cause the follower to revolve in the *same* direction as the driver, and would not produce any change in the proportion between their angular motions, the radii AC and BC being unchanged.

95. *Practical objection to the above solution.*—It is evident that the above solution fails if the surfaces of the wheels rub smooth, so that the motion becomes partly one of sliding and partly one of rolling contact; and also that it will fail if the centres A and B are slightly displaced, since then the contact ceases: one method, in common use, of obviating this objection is to pass a powerful band of leather tightly over the wheels; this method is commonly used when the centres A and B are so considerable a distance apart that the wheels would be inconveniently large if in immediate contact; the most effectual means, and the only one with which we are here concerned, is to cut teeth on the circumferences of the wheels; when this is properly done the uniform revolution of the wheel A can be made to communicate a uniform revolution to the wheel B. The problem we are to solve is therefore twofold:—

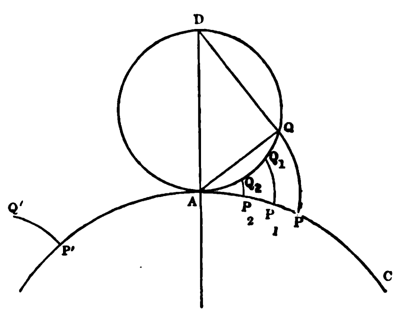
(1.) To determine the form that must be given to teeth of wheels, in order that any uniform motion of the driver round its axis shall communicate to the follower a *uniform* motion round its axis.

(2.) As this cannot be done without causing the teeth of the one wheel to *slide* over those of the other, it is required to determine what amount of work is lost by the friction of the teeth when work is transmitted from one axle to the other.

The limits of the present work will not allow us to do more than give one solution of the former question, and an approximate solution of the second. Readers who desire further information on this very important subject, will be able to obtain it by reference to Mr. Willis's Principles of Mechanism, and to Mr. Moseley's Mechanical Principles of Engineering: the former work only treats of the question of *form;* the latter also contains a very full discussion of the question of *force.*

96. *Definition and properties of the epicycloid.*—If a circle carrying on its circumference a pencil-point be made to roll on the outside of the circumference of a fixed circle, the point will trace out a curve called an *epicycloid:* the fixed circle is called the *base;* the moveable circle is called the *generating circle.* Thus if Q is a point on the generating circle ADQ, and APC is the base or fixed circle, then if Q were in contact with APC at P, the point Q will trace out the epicycloid P Q.

Fig. 123.



(*a.*) It is evident that the length of the arc AQ equals that of the arc AP.

(*b.*) It is evident that the point Q is at the instant moving in a circle of which the centre is A, and radius AQ, so that the line AQ is the normal to the epicycloid at the point Q, and if DQ be joined that line is a tangent to the curve at Q.

(*c.*) It is evident that the form and dimensions of the curve are independent of the particular point Q occupies on the generating circle, so that if we take a succession of points Q, Q_1, Q_2... on the generating circle, and describe with them a succession of epicycloids QP, $Q_1 P_1$, $Q_2 P_2$... they will all be exactly like one another, and if P′Q′ be any epicycloid described on the same base with the same generating circle as the others, it too will be exactly like the rest: if we now suppose all the former to remain fixed, and the circle P′AC to revolve round its centre, carrying P′Q′ with it, then when P′ comes to P_2, the curve P′Q′ will fall upon $P_2 Q_2$, and in like manner on $P_1 Q_1$ and on PQ.

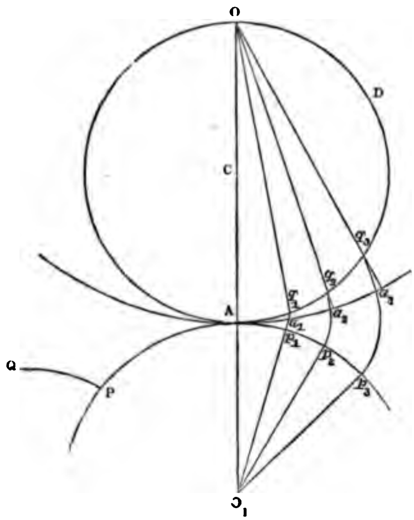
Proposition 21.

An epicycloidal tooth can be made to work correctly with a straight tooth.

Let PQ be the tooth described on the base AP, the centre of which is O_1, by a circle whose diameter is AO; suppose the base to revolve round O_1 and let the tooth assume successively the positions $p_1 q_1$, $p_2 q_2$, $p_3 q_3$.... cutting the circle ADO in points q_1, q_2, q_3, then since the straight lines Oq_1, Oq_2, Oq_3.... touch the epicycloid in the points q_1, q_2, q_3... it is plain that a straight line whose length is OA, and which is moveable round O, will, if driven by the tooth, come successively into the positions Oa_1, Oa_2, Oa_3... passing through the points q_1, q_2, q_3....respectively. Now if we suppose the angles $AO_1 p_1$, $p_1 O_1 p_2$.... to be equals,

the arcs A p_1, $p_1 p_2$, $p_2 p_3$.... are equal, and therefore (Art. 96 (*a*)) the arcs A q_1, $q_1 q_2$, $q_2 q_3$... are equal, and the angles they subtend at C will be equal, and their halves

Fig. 124.



will be also equal, *i.e.* the angles AOa_1, a_1 Oa_2, a_2 Oa_3, are equal; so that if the circle PAO$_1$ moves with a uniform angular motion, it will communicate a uniform angular motion to a straight line AO moveable about the point O, *i. e.* the straight line works truly with the epicycloidal tooth.

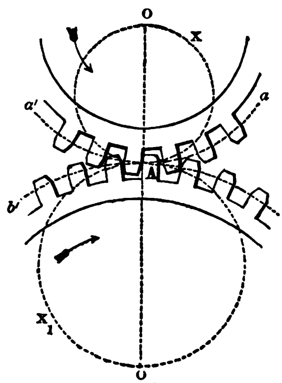
Ex. 494.—If with centre O and radius OA a circle is described, show that if this circle works with AP by friction, any one of its radii will have the same angular velocity as if it had been driven by the tooth PQ.

97.—*Practical rule for the form of teeth.**—Let O O$_1$ be the centre of the two toothed wheels; draw the line of

* This rule, though not the *best*, is very generally employed in practice. See Willis, p. 106.

centres O, O_1, when the point of contact of any two teeth is on the line of centres let it be at A; with centres O and O_1 and radii O A and O_1 A respectively describe circles, a A a', b A b'; these are called the *pitch circles* of the respective wheels, *i.e.* the two circles which rolling by friction would move with the same angular motions as the wheels. Now if there are to be m teeth in the wheel O, there must be m_1 in the wheel O_1, where m_1 is given by the proportion

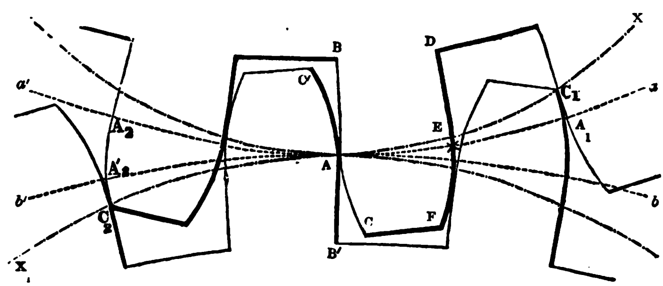
Fig. 125.



$$\text{O A} : \text{O}_1\text{A} :: m : m_1$$

Divide the circumference of a A a' into m equal parts, of which parts let A A_1 be one; the chord of this arc is

Fig. 126.



called the *pitch* of the wheel; divide it into two (nearly) equal parts, of these A E (the smaller) is the breadth of a tooth, and E A_1 the space between two teeth; then the flanks BA, DE of a tooth (*i. e.* the parts of its outline within the pitch circle) are straight lines converging to the centre O; and the faces of the tooth A C, E F (*i.e.* the parts of its

outline on the outside of the pitch circle) are portions of epicycloids described on the pitch circle as a base by a generating circle whose diameter equals the radius of the pitch circle of the wheel with which it is to work, viz. $O_1 A$. The teeth of the wheel O_1 are cut upon the same principle; the circumference of the pitch circle $b \, A \, b'$ is divided into m_1 equal parts, and this as before divided into a tooth and a space; the flanks of the teeth converge to O_1, the faces are epicycloids described on the pitch circle as a base by a generating circle whose diameter equals the radius O A. That the two wheels thus constructed will work truly, follows immediately from Prop. 21; thus, if the wheel O revolves uniformly, the tooth B A C driving the tooth B′A C′, then the epicycloid AC will cause the straight line A B′, and therefore the wheel O_1, to revolve uniformly: on the other hand, if the wheel O_1 moving with a uniform motion drives O, the epicycloid A C′ will cause the straight line A B, and therefore the wheel O, to revolve uniformly. This is of course true whether the wheels move in the same or in contrary directions to those indicated by the arrow-heads in fig. 125. In order to prevent the *locking* of the teeth, it is usual to make A E less than $E A_1$ by $\frac{1}{11}$th of the pitch $A A_1$; and to cut the space A B′ deeper than the perpendicular length of the tooth in such a manner that the distance from C to the centre is less than the distance from B′ to the same centre by $\frac{1}{10}$th of the pitch $A A_1$; if however the workmanship is very good, the differences can in both cases be made smaller.

The rule for determining the length of the teeth commonly adopted by millwrights, is to make the length of the tooth beyond the pitch circle (*i.e.* A C or A C′) equal to $\frac{3}{10}$ths of the pitch.* This rule is, however, a very bad one;

* Willis's Principles of Mechanism, p. 98. The rule which follows is given both by Mr. Moseley, Mechanical Principles, p. 267, and by Gen. Morin, Aide-Mémoire, p. 280.

the following, though not perhaps the best, is very much better. Suppose O to be the driver, and suppose a pair of teeth to be in contact on the line of centres, the face of the next tooth should be so long that its extreme point C_2 should just be on the circumference of the generating circle $A X_1$, as shown in the figure; the length of the tooth of the follower is determined by a similar rule; the extreme point of the following tooth C_1 should (under the same circumstances) be on the circumference of the generating circle A X O. The reason of this rule is as follows: it may be considered that when the wheels are in motion that pair will bear the whole or nearly the whole strain which at any instant will be the next to go out of contact; so that, the construction above being employed, the one pair of teeth is just going out of contact when the next pair comes to the line of centres, and consequently the working strain is not thrown upon any pair of teeth until it comes to the line of centres; but it appears that practically the friction between a pair of teeth is very much more destructive when they are in contact before the line of centres than when in contact behind the line of centres; by following, therefore, the rule above given, the friction between any pair of teeth is diminished. (Compare Ex. 516.)

In practice the teeth of a wheel are all cut from a pattern; in constructing a pattern the epicycloidal curve may be drawn by the actual rolling of a circle of the proper size; or an approximation may be obtained by means of circular arcs. Rules proper for this purpose will be found in Mr. Willis's Treatise above referred to.

Ex. 495.—To determine the radius of the pitch circle of a wheel which shall contain n teeth of given pitch a. *Ans.* $r = \dfrac{a}{2 \sin \frac{80}{n}}$

Ex. 496.—If a wheel of m teeth drives another of n teeth; then if the driver makes p revolutions per minute the follower will make $\frac{mp}{n}$ revolutions per minute.

Ex. 497.—There are three parallel axes A, B, C; A makes p revolutions per minute, it carries a wheel of m_1 teeth which works with a wheel of n_1 teeth on B; B also carries another wheel of m_2 teeth which works with a wheel of n_2 teeth on C; show that C makes $\frac{m_1}{n_1} \cdot \frac{m_2}{n_2} \cdot p$ revolutions per minute.*

Ex. 498.—A winding engine is worked in the following manner; a steam engine causes a crank to make 30 revolutions per minute; the axle of the crank has on it a wheel containing 36 teeth, which works with a wheel containing 108 teeth; the latter wheel is on the same axle as the drum, which is 5 ft. in radius; determine the number of feet per minute described by the load. *Ans.* 314 ft.

98. *The Hunting Cog.*—If wheels have to do heavy work, and the precise proportion between the velocities not of great importance, an additional tooth—called a *hunting cog*—is introduced into one of the wheels, so that the same pair of teeth may seldom work together; by this means they are kept from wearing unequally; for instance, if in the last Example we denote the teeth of the driver by the successive numbers 1, 2, 3, . . . 36, and the teeth of the follower by the successive numbers 1, 2, 3, . . . 108. Then in every revolution 1 will work with 1, 37, and 73; 2 will work with 2, 38, and 74; and 36 will work with 36, 72, and 108. If now we introduce a hunting cog into the driving-wheel, so that it contains 37 teeth, then on the first revolution 1 will work with 1, 38, and 75; in the next revolution with 4, 41, and 78, in the third with 7, 44, and 81, and not until the 38th revolution will it work with 1 again.

Ex. 499.—If in the last Example a "hunting cog" were introduced into the driver so that it contains 37 teeth, determine the number of feet per minute the load will now travel. *Ans.* 323 ft.

Ex. 500.—If in Example 497 there are $k+1$ axles and the drivers contain

* The above arrangement is to be found in most *cranes;* if the student is not acquainted with the arrangement of a train of wheels he will do well to examine a good crane such as is to be seen at most railway stations: the train of wheels in a clock is also a good example, but cannot commonly be studied without taking the clock to pieces.

m teeth, and the followers contain n teeth a-piece, show that the number of revolutions made by the last axle will be $\left(\frac{m}{n}\right)^k p$.

Ex. 501.—If in the last Example it is required to multiply the number of revolutions 200 times, how many axles must we use, (1) if we take $m = 2n$; (2) if we take $m=4n$; (3) if we take $m=6n$, and determine the number of teeth employed, in each case using the nearest whole numbers.

Ans. Axes (1) 8. (2) 4. (3) 3.
Teeth (1) $54n$. (2) $20n$. (3) $21n$.

Ex. 502.—If each driver has m teeth, and each follower n teeth, and if M is the total number of teeth in the train, and if the last axle makes q revolutions while the first axle makes one revolution, show that

$$q = \left(\frac{m}{n}\right)^{\frac{M}{m+n}}.$$

Ex. 503.—In the last Example show that for a given value of M we shall obtain the greatest value of q by making $m = 3{\cdot}59 \,.\, n$ nearly.

[It is easily shown that $\log\left(\frac{m}{n}\right) = 1 + \frac{n}{m}$, whence the result stated.]

Ex. 504.—In the case of a pair of wheels with epicycloidal teeth show that the space through which the surfaces of each pair of teeth slide one upon the other while in contact and after passing the line of centre is approximately represented by the formula $\frac{2\pi r}{n}\left(\frac{\pi}{n}+\frac{\pi}{n_1}\right)$ or $\frac{2\pi r_1}{n_1}\left(\frac{\pi}{n}+\frac{\pi}{n_1}\right)$ where r and r_1 are the radii of the driver and follower respectively, and n and n_1 the number of teeth in those wheels respectively.

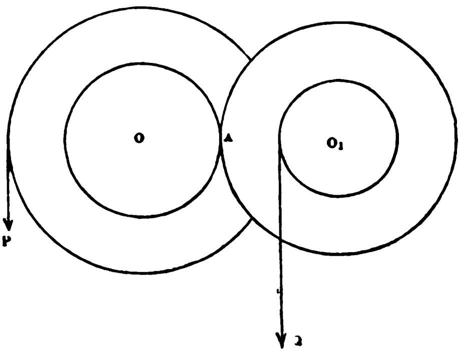
[Referring to fig. 126 it is evident that the pair of teeth just going out of contact touch at C_2; it is also evident that the two points A_2 and A'_2 were in contact at A, so that the space through which the surfaces have slidden over each other is $A_2\,A'_2$, which is very nearly equal to the sum of the versed sines of the *arcs* AOA_2 and $AO_1\,A'_2$, *i.e.* to $r \text{ vers } \frac{2\pi}{n} + r_1 \text{ vers } \frac{2\pi}{n_1}$; whence the value assigned in the question.]

Ex. 505.—A weight P balances a weight Q under the following circumstances; P is tied to a rope which is wrapped round an axle whose radius is p; Q is tied to a rope which is wrapped round an axle whose radius is q; to the former is attached a concentric rough wheel, whose radius

* It would appear from this that the best proportion between the number of teeth in driver and follower for multiplying velocity is 1 : 4. This result is due to Dr. Young, Lectures, vol. ii. p. 56. Mr. Willis remarks that the rule is not of much practical value, Principles, p. 218.

is r, to the latter in like manner a concentric rough wheel, whose radius is r_1; these two wheels are in contact on the line of centres so that $r+r_1$ equals $O\,O_1$, show that if we neglect the magnitude of the axes and the rigidity of the cords, we shall have

Fig. 127.



$$P=Q\,\frac{q}{p}\cdot\frac{r}{r_1}.$$

[The arrangement described in the above example is represented in the annexed diagram; it is evident that the rough wheels act on each other by means of a mutual action through the point A.]

Ex. 506.—In the last Example if we suppose the separate wheel and axles to turn round axes whose radii are ρ and ρ_1 respectively and the limiting angles of resistance between them and their bearings to be ϕ and ϕ_1, show that when P is on the point of overcoming Q we have the following relation (neglecting the rigidity of cords, and the weights of the wheel and axles)

$$P\,(p-\rho\sin\phi)\,(r_1-\rho_1\sin\phi_1)=Q\,(q-\rho_1\sin\phi_1)\,(r+\rho\sin\phi).$$

Ex. 507.—If in the last Example, besides the frictions on the axes, we take into account the weights W and W_1 of the wheel and axles, determine the relation between P and Q.

Ex. 508.—If in the last Example we neglect powers and products of $\frac{\rho\sin\phi}{p}$, $\frac{\rho\sin\phi}{r}$, $\frac{\rho_1\sin\phi_1}{q}$, $\frac{\rho_1\sin\phi_1}{r_1}$, show that the number of units of work that must be done in order to raise a weight of Q lbs. through a space of s ft. is given by the formula

$$U=Qs\left\{1+\left(\frac{1}{p}+\frac{1}{r}\right)\rho\sin\phi+\left(\frac{1}{q}-\frac{1}{r_1}\right)\rho_1\sin\phi_1\right\}$$
$$+\frac{r_1 s}{q}\left\{W\,\frac{\rho\sin\phi}{r}+W_1\,\frac{\rho_1\sin\phi_1}{r_1}\right\}.*$$

Ex. 509.—In the last Example if we suppose the rough wheels to be replaced by a pair of toothed wheels whose pitch circles have the same radii

* If P instead of being a weight were a pressure acting vertically upward, it is easily shown that the third term of this equation is

$$Qs\left(\frac{1}{q}+\frac{1}{r_1}\right)\rho_1\sin\phi_1.$$

as the wheels; then if the wheel O contains n teeth, and the wheel O_1 contains n_1 teeth, show that when Q is raised through a space of s ft. the work lost by the friction of the teeth is approximately represented by the formula $\mu\, Q\, s\left(\frac{\pi}{n}+\frac{\pi}{n_1}\right)$, where μ is the coefficient of friction between the teeth.

[If the wheel $O_1 A$ revolves through an angle $\frac{2\pi}{n_1}$ the space through which the surfaces of the driving and driven teeth slide is $\frac{2\pi r_1}{n_1}\ \frac{\pi}{n}+\frac{\pi}{n_1}\Big)$ and therefore, supposing R, *the mutual pressure, to continue constant* during the contact of the teeth, the number of units of work expended on friction equals $\mu\ R\ \frac{2\pi r_1}{n_1}\left(\frac{\pi}{n}+\frac{\pi}{n_1}\right)$. Now, approximately, $Rr_1 = Qq$, and therefore the work expended on one pair of teeth equals $\mu\, Q\frac{2\pi q}{n_1}\left(\frac{\pi}{n}+\frac{\pi}{n_1}\right)$; but $\frac{2\pi q}{n_1}$ is the space through which Q is raised during the action of one pair of teeth, and the same being true of every pair of teeth we obtain the result stated in the question. Of course the addition of the expression contained in the present question to that obtained in the last is the correct approximate formula for the work expended in raising a weight through the intervention of a pair of toothed wheels.]

Ex. 510.—A pressure P acting at the end of an arm O A, two feet long, causes the toothed wheel O B to make 10 turns per minute; this wheel working with the wheel O_1 B turns the drum O_1 C and raises the weight Q; given that P does at the point A 330000 units of work per minute, determine approximately the weight Q that will be raised by the drum, having given the radius of O B to be 1 foot, O_1 B to be 3 feet, the number of teeth in O B to be 40, and the radius of the drum 5 feet; the teeth, axles and bearing are all of cast iron without unguents; the radii of the axles are 3 in., the weight of the axles and appendages of O are 3600 lbs., and that of O_1 being 5400 lbs. *Ans.* (1) 2724 lbs.

[See Note to Ex. 508.]

Ex. 511.—Show that in a train of p pairs of wheels and pinions* the work lost by friction between the teeth is given by the formula

$$\mu\, Q\, s\, \pi\left\{\frac{1}{n_1}+\frac{1}{n_2}+\frac{1}{n_3}+\dots+\frac{1}{n_{2p}}\right\}$$

* When a small wheel drives a large one the former is frequently called a pinion and the latter a wheel.

where $n_1, n_2, n_3 \ldots n_{2p}$ are the number of teeth in the successive wheels and pinions.

Ex. 512.—There is a train of p equal pairs of wheels and pinions; the numbers of teeth are such that the last axle revolves m times faster than the first; show that if U is the number of units of useful work yielded, the work lost by the friction between the teeth is represented by the formula

$$\frac{\mu U \pi p}{n}\left(1+m^{\frac{1}{p}}\right)$$

where n is the number of teeth in each wheel.

*Ex. 513.—If it is required to make the last axle move m times faster than the first, show that the loss of work is least when p, the number of pairs of wheels and pinions, is given by the formula

$$m^{-\frac{1}{p}} + \log_e m^{-\frac{1}{p}} + 1 = 0.$$

Ex. 514.—If in the last Example it is required to multiply the velocity 100 times, show that the proper number of pairs of wheels and pinions is 3 or 4, *i.e.* show that the equation in the last Example gives a value of p between 3 and 4; and determine the number of teeth employed in each case if the first pinion have 20 teeth, using the nearest whole numbers.

Ans. (1) 339. (2) 332.

Ex. 515.—If in the pair of wheels already described (Art. 97) all but a single pair of teeth be cut away, so that the remaining pair act on each other while the wheel O moves through an angle $\frac{2\pi}{n}$ before coming to the line of centres, and also while it moves through an equal angle after having passed the line of centres, and if we suppose P and Q to act on the pitch circles of their respective wheels, show that when the point of contact is in such a position that the wheel O has to revolve through an angle θ before the point of contact comes to the line of centres we have

$$P\left\{r_1-(r+r_1)\tan\theta\tan\phi\right\}=Q\,r_1$$

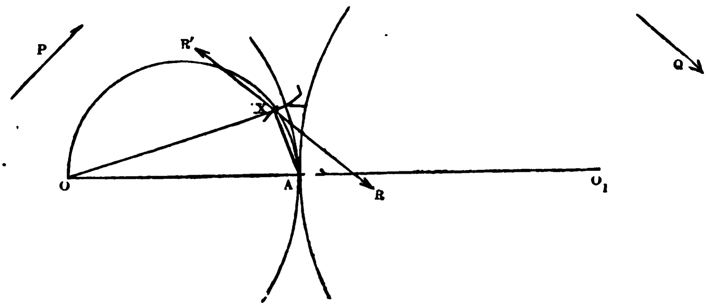
and that when the point of contact is so situated that the driver has revolved through an angle θ from the line of centres we have

$$P\,r=Q\left\{r+(r+r_1)\tan\frac{r\theta}{r_1}\tan\phi\right\}$$

[If in the accompanying figure X is the point of contact of the teeth before they come to the line of centres, that point X will be on the circum-

ference of a circle whose diameter is O A; if then we draw a line RR′ such that the angle RXA equals ϕ, this will be the line of the mutual action of

Fig. 128.



the teeth; remembering that the angle AOX equals θ it is easily shown that the perpendiculars on RR′ from O and O_1 are respectively equal to

$$r \cos\theta \cos\phi$$

$$\text{and} \quad (r+r_1)\cos(\theta+\phi) - r\cos\theta\cos\phi$$

whence the first equation is obtained; the second is obtained in a similar manner, by determining the relation between P and Q when the follower has revolved through an angle θ' which will be found to be

$$Pr = Q\,\{r + (r+r_1)\tan\theta'\tan\phi\}$$

whence we obtain the answer.]

Ex. 516.—If the driver be not greater than the follower, show from the equations of the last example, that for a given value of Q, the value of P is greater when the driving tooth is in a given position *before* it comes to the line of centres than when it is in a corresponding position after having passed the line of centres.

[Expand the values of P in terms of θ and ϕ, omitting higher powers than the fourth.]

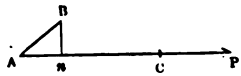
Ex. 517.—If AB be any diameter of a circle APB; if C be any point taken in the prolongation of AB (so that B is between A and C), and if AP, BP, CP be joined, show that

$$BC = AC \tan PAB \tan BPC$$

and hence explain the action of the pressures which produce the result which follows from the first equation in Ex. 515, viz. that when $r_1 = (r+r_1)\tan\theta\tan\phi$ the pressure P must be infinitely large to bring Q into the state bordering on motion.

99. *Definition of the virtual velocity and virtual moment of a pressure.*—Let P be a pressure acting at the point A along the line A P, and let it be represented by A C (Art. 27). Suppose P's point of application to be shifted through an indefinitely small space to B; draw

Fig. 129.



B n at right angles to A C or C A produced, and let A n be denoted by p, which is commonly reckoned positive when n falls between A and C, and negative when it falls on C A produced; then p is called the virtual velocity of P, and P p its virtual moment. It will be observed that the virtual moment of P is the work done by that pressure when its point of application receives an indefinitely small displacement A B.

100. *The principle of virtual velocities.*—This principle is as follows: If a system of pressures in equilibrium acts on any machine which receives any small displacement—consistent with the connection of the parts of the machine—the algebraical sum of the virtual moments of the pressures will equal zero. It will be remarked that if the principle laid down at the beginning of this chapter (Art. 90) be accepted as fundamental, it includes the principle of virtual velocities as a particular case.*

If P_1, P_2, P_3 . . . . . are the separate pressures, and p_1, p_2, p_3 . . . . their virtual velocities, the principle is expressed algebraically by the following equation, which is commonly called the equation of virtual velocities:

$$P_1 p_1 + P_2 p_2 + P_3 p_3 + \dots = 0.$$

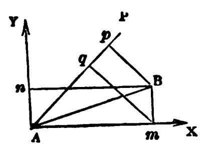
Of course those pressures which do no work, as explained in Art. 92, will disappear from the above equation. Although from the method here followed the discussion of

* Compare Mr. Rankine's Appl. Mech. p. 479, and Morin, Notions Fond., Art. 30.

this principle is quite isolated from the rest of the work, it will be interesting to verify its truth in certain particular cases.

Ex. 518.—If X and Y are the rectangular components of a pressure P, to show that the virtual moment of P equals the sum of the virtual moments of X and Y.

Fig. 130.



Let A be the point of application of P and let A be transferred to B; complete the rectangle mn, and draw Bp and mq at right angles to AP; then Ap, Am, An, are the virtual velocities of P, X, and Y, and we have to prove that

$$P\,.\,Ap=X\,.\,Am+Y\,.\,An.$$

Let XAP be denoted by θ, then it is evident that

$$Ap=Aq+qp=Am\,.\cos\theta+An\sin\theta$$
$$\therefore\ P\,.\,Ap=Am\,.\,P\cos\theta+An\,.\,P\sin\theta$$
$$\text{or } P\,.\,Ap=X\,.\,Am+Y\,.\,An\ ....\ (1)$$

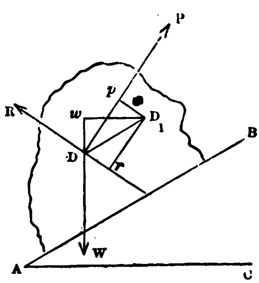
If P had acted in the contrary direction X, Y, and P would have been in equilibrium; the virtual moment of P would be negative; and (1) would become the equation of virtual velocities.

Ex. 519.—In the last Example suppose that P balances X and Y, and suppose its point of application to be transferred in a direction at right angles to AP, verify the equation of virtual velocities.

[It must be remembered that in this case P's virtual moment equals zero. (Art. 92 *b*.)]

Ex. 520.—Show that the principle of virtual velocities is true in the case of a body in the state bordering on motion up an inclined plane, when a small motion is given to it either up or down the plane.

Fig. 131.



[Draw the figure as in Ex. 251, then, if the motion takes place up the plane, D will be transferred to a point D_1 along a line DD_1 parallel to AB; let fall from D_1 perpendiculars on the directions of the pressures, viz. D_1w, D_1p, D_1r, then Dw, Dp, Dr, are the virtual velocities of the pressure, and of them Dp is positive and the others negative; the equation of virtual velocities therefore becomes

$$P\,.\,Dp=W\,.\,Dw+R\,.\,Dr$$

and this the student is required to prove.]

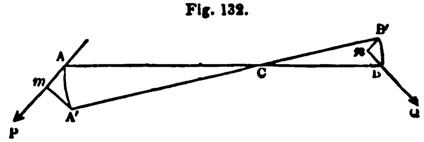
Ex. 521.—Verify the principle of virtual velocities in the last case, as-

suming that the plane (and with it the body) is so moved that D describes a straight line at right angles to DR.

Ex. 522.—Verify the principle of virtual velocities in the case of two pressures in equilibrium on a straight bar capable of turning round a fixed point.

[Let P and Q be the pressures which balance on the rod AB round the fixed point C; suppose the rod to turn through a small angle and to come into the position A′B′; draw A′m at right angles to AP and B′n at right angles to BQ, then Am is the virtual velocity of P and Bn of Q, the latter being negative; also the virtual moment of the reaction of C is zero (Art. 92); the equation to be proved is therefore

Fig. 132.



$$P \cdot Am = Q \cdot Bn.$$

The student must remember that AA′m and BB′n are ultimately right-angled triangles.]

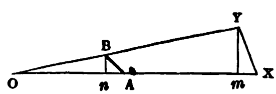
Ex. 523.—Verify the principle in the case of two parallel pressures P and Q which keep a beam at rest round a rough axle of finite dimensions (as in Ex. 307), the motion being given to the beam round the axle.

Ex. 524.—In the last Example how would it be possible to move the system so that the reaction R should disappear from the equation of virtual velocities.

Ex. 525.—In Ex. 523 show that when the axle is smooth the reaction will disappear from the equation of virtual velocities.

101. *Proof of the principle of virtual velocities.*—The following proof applies to the case of any system of pressures acting on a single rigid body in one plane, in which the displacement is supposed to be made: it can be easily extended so as to include the pressures that act on any machine.

Fig. 133.



Lemma.—Let A and X be any two points in a given line, let the line be transferred to any consecutive position O Y, so that A comes to B and X to Y, then if B Y equals A X, and if B n and Y m are drawn at right angles to A X, the line A n will ultimately equal X m.

For $n\,m$ equals B Y cos O, *i.e.* it ultimately differs from B Y, and therefore from A X, by a small quantity of the second order; take away the common part A m, then A n and X m ultimately differ by a small quantity of the second order; but they are themselves of the first order, and therefore are ultimately equal.

Cor.—Hence if a pressure acts along a certain line, and if two points in the line are rigidly connected, its virtual velocity will be the same at whichever point we suppose it to act; also if there are two equal and opposite pressures, their virtual moments will be equal and have contrary signs, whether we suppose them to act at the same point or at two rigidly connected points.

(a) If a number of parallel pressures act at right angles to a given line, and at points in that line, the virtual moment of the resultant will equal the sum of the virtual moments of the pressures.

Let $X_1, X_2, X_3 \ldots$ be the pressures; when the line on which they act has been displaced let it cut its first position in a point O, and make with it a small angle θ, also let $y_1, y_2, y_3 \ldots$ be the distances of the points of application of the pressures from O. Then we have (Prop. 14)

$$X y = X_1 y_1 + X_2 y_2 + X_3 y_3 + \ldots$$

where X is the resultant, and y the distance of its point of application from O; and therefore

$$X y\,\theta = X_1 y_1 \theta + X_2 y_2 \theta + X_3 y_3 \theta + \ldots \qquad (1)$$

Now ultimately the perpendicular heights of the second positions of the points of application above their first position are the virtual velocities of the several pressures, and these are ultimately $y\,\theta$, $y_1\,\theta_1$, $y_2\,\theta_2$, $y_3\,\theta_3$, ... and therefore equation (1) means that the virtual moment of the resultant equals the sum of the virtual moments of the pressures.

Cor.—It follows from the lemma, that if we suppose the pressures to act at any points in their direction, the above rule will be true provided those points are rigidly connected.

(*b*) If any pressures $P_1, P_2, P_3 \ldots$ act in one plane, and have a resultant R, the points of application being rigidly connected, the virtual moment of R will equal the sum of the virtual moments of $P_1, P_2, P_3 \ldots.$ or

$$Rr = P_1 p_1 + P_2 p_2 + P_3 p_3 + \ldots.$$

For let the pressures be resolved parallel to two rectangular co-ordinates; then (Ex. 518) the virtual moment of P_1 equals the sum of the virtual moments of its two components, and similarly of $P_2, P_3 \ldots.$; let X be the resultant of the one set of components, and Y that of the other, and let their directions intersect in a certain point (A), R will act through that point, and will be the resultant of X and Y, and therefore Rr will equal the sum of the virtual moments of X and Y supposed to act at A; but the virtual moment of X equals the sum of the virtual moments of one set of components, and Y that of the other (by Cor. to *a*), and the sum of these equals $P_1\, p_1 + P_2\, p_2 + P_3\, p_3 \ldots.$

(*c*) If $P, P_1, P_2, P_3 \ldots.$ are pressures acting on a body and in equilibrium, the sum of their virtual moments will equal zero.

For let R be the resultant of $P_1, P_2, P_3, \ldots.$ then provided the forces act in one plane—the only case we are at present concerned with—

$$P_1 p_1 + P_2 p_2 + P_3 p_3 + \ldots. = Rr.$$

But R is equal and opposite to P, therefore by the Cor. to the lemma

$$Pp + Rr \qquad = 0$$

$$\therefore \text{(by addition) } Pp + P_1 p_1 + P_2 p_2 + P_3 p_3 + \ldots. = 0$$

Q. E. D.

PART II.

DYNAMICS.

CHAPTER I.

INTRODUCTORY.*

102. *Velocity.*—Before considering *force* as the cause of *velocity* or of *change* of *velocity* it will be necessary to define accurately the means of estimating the magnitude of velocities.

Def.—A body moves uniformly or with a uniform velocity when it passes over equal spaces in equal times.

The units of space and time commonly employed are feet and seconds†: and whenever a body is said to be moving with any particular velocity, *e. g.* 5 or 6, this will always mean with a velocity of 5 or 6 feet per second.

Def.—When a body moves with a variable velocity, that velocity is measured at any instant by the number of units of space it would pass over in a unit of time *if it continued to move uniformly from that instant.*

It will be seen from the definition that variable velocity

* The student is particularly recommended to make himself thoroughly master of this Chapter before proceeding further.

† To prevent mistake it may be stated that the time referred to is *mean solar time.*

is measured in a manner that exactly falls in with the ordinary way of speaking: thus when we say that a train is moving at the rate of 40 miles an hour, we mean that if it were to keep on moving uniformly for an hour, it would pass over 40 miles: again if we were to drop a small heavy body, we should find that at the end of a second it is moving at the rate of about 32 feet per second, or as it is commonly stated, it acquires in a second a velocity 32, meaning that if it were to move uniformly from the end of that second it would pass over 32 feet in each successive second.

103. *Relation between uniform velocity, time, and space.* —In the case of a body moving with a uniform velocity, it is evident that the number of feet (s) passed over in t seconds must be t times the number of feet passed over in one second (v)

$$\therefore s = vt.$$

The space s can, of course, be represented geometrically by the area of a rectangle whose sides severally represent on the same scale the velocity and the time.

Ex. 526.—A body moves uniformly over $2\frac{1}{2}$ miles in half an hour, determine its velocity. *Ans.* $7\frac{1}{3}$.

Ex. 527.—A body moves at the rate of 12 miles an hour, determine its velocity. *Ans.* $17\frac{3}{5}$.

Ex. 528.—The equatorial diameter of the earth is 41,847,000 ft., and the earth makes one revolution in 86,164 seconds, determine the velocity of a point on the earth's equator. *Ans.* 1526.

Ex. 529.—A body moves with a velocity 12; how many miles will it pass over in one hour? what would be its velocity if we used yards and minutes as units instead of feet and seconds? *Ans.* (1) $8\frac{2}{11}$. (2) 240.

104. *The velocity acquired by falling bodies.*—It appears as a result of the most careful experiment that at any given point of the earth's surface, a body falling freely in vacuo acquires at the end of every second a certain con-

stant additional velocity *: this velocity is slightly different at different places, but is always the same at the same place, and never differs greatly from 32; so that if at any instant the falling body have a velocity V, it will have at the end of the next second a velocity of $V+32$. This additional velocity is commonly called the *accelerating* force of gravity, and is denoted by the letter g:—in all the following examples it will be assumed that g equals 32, unless the contrary is specified.

From what has been said it is plain that if a body is let fall, it acquires a velocity g at the end of the first second, $2g$ at the end of the second second, $3g$ at the end of the third second, and so on: consequently, if v is the velocity acquired at the end of t seconds, we shall have

$$v = gt.$$

By the same reasoning it appears that if the body is thrown *downward* with a velocity V, and if v is its velocity after falling for t seconds, then

$$v = V + gt.$$

Moreover it appears, when a body is thrown *upward* so as to move in a direction opposite to that in which gravity acts, that then it loses in every second a velocity g; consequently in this case

$$v = V - gt.$$

Ex. 530.—A body is let fall for 7 seconds, with what velocity is it moving at the end of that time? *Ans.* 224.

Ex. 531.—If a body is let fall, how long will it take to acquire a velocity of 200 ft. per second? *Ans.* $6\frac{1}{4}$.

Ex. 532.—A body is projected downward with a velocity of 80 ft. per second; determine the velocity it will have at the end of 5 seconds, and the number of seconds that must elapse before a velocity equals twice its initial velocity? *Ans.* (1) 240. (2) $2\frac{1}{2}$ sec.

Ex. 533.—A body is thrown downward with a velocity of 160 ft. per

* It may be remarked that the difference between the velocities with which a feather and a bullet descend is entirely due to the resistance of the air.

second, determine its velocity at the end of 4 seconds, and the number of seconds in which a body that is merely dropped would acquire that velocity. *Ans.* (1) 288. (2) 9 sec.

Ex. 534.—A body A is projected downward with a velocity of 160 feet per second; at the same instant another body B is projected upward with an equal velocity; determine how much faster A will be moving than B at the end of 4 seconds. *Ans.* 9 times.

Ex. 535.—A body is thrown upward with a velocity of 96 feet per second; with what velocity will it be moving at the end of 4 seconds?

[The formula gives -32, *i. e.* it will be moving *downward* with a velocity of 32 feet per second.]

Ex. 536.—In the last case how long will it take the body to reach the highest point?

[It will be at the highest point when $v=0$, *i. e.* after 3 seconds.)

Ex. 537.—A body is at any instant moving *upward* with a given velocity V, show that it will be moving downward with an equal velocity after $\frac{2V}{g}$ seconds; and that it will reach its highest point after $\frac{V}{g}$ seconds.

105. *The space described in a given time by a falling body.*—It admits of proof that if a body is allowed to fall freely from rest for t seconds the number of feet (s) which it will pass over is given by the formula

$$s = \tfrac{1}{2}gt^2.$$

If, however, it is *thrown downward* with a velocity V, we shall have

$$s = Vt + \tfrac{1}{2}gt^2$$

and if *upward* with a velocity V, it will, at the end of t seconds, be s feet above the point of projection, where

$$s = Vt - \tfrac{1}{2}gt^2.$$

Ex. 538.—How many feet will be described in 4 seconds by a body that moves freely from rest under the action of gravity? *Ans.* 256 ft.

Ex. 539.—Through how many miles would a body falling freely from rest descend in one minute? *Ans.* $10\frac{10}{11}$ mi.

Ex. 540.—A body is projected downward with a velocity of 20 ft. per second; how far will it fall in $1\frac{1}{2}$ seconds? *Ans.* 66 ft.

Ex. 541.—A body is projected upward with a velocity of 100 ft. per second; how high will it have ascended in three seconds? *Ans.* 156 ft.

Ex. 542.—Show that the greatest value of $Vt-\frac{1}{2}gt^2$ is found by making $t=\frac{V}{g}$.

[Compare this result with Ex. 537.]

Ex. 543.—If a body is projected *upward* with a velocity of 96 ft. per second, where will it be at the end of 7 seconds, and what will be the whole space it will have described?

Ans. (1) 112 ft. below the point of projection. (2) 400 ft.

Ex. 544.—A body is projected upward with a velocity of 100 ft. per second, determine where the body will be, with what velocity the body will be moving and in what direction at the end of 4 seconds.

Ans. (1) 144 ft. above the point of projection. (2) 28 ft. per sec. downward.

Ex. 545.—A body is projected upward with a velocity V, show that it will return to the point of projection after $\frac{2\,\mathrm{V}}{g}$ seconds.

[Compare this result with Ex. 537.]

106. *Relation between velocity acquired and space passed over by a falling body.*—The above relations between the velocity (v) which the body has at the end of a time (t) and between the space (s) which it describes on the same time (t) enable us to determine the relation between v and s; thus if the body is simply let fall we have

$$v = gt$$

and

$$s = \tfrac{1}{2}gt^2$$

whence

$$v^2 = 2gs$$

an equation which gives the velocity acquired in falling from rest through s feet. From the corresponding equations the reader will easily deduce the following

$$v^2 = \mathrm{V}^2 + 2gs$$
$$v^2 = \mathrm{V}^2 - 2gs.$$

The former of these gives the velocity (v) which the body has after falling through s feet when it was thrown down with a velocity V; the latter the velocity (v) which it has when it is s feet above the point from which it was thrown up with the velocity V. Whether the direction of this velocity (v) is upward or downward must be determined by other considerations.

Ex. 546.—If a body is thrown upward with a velocity V, show that it will ascend through $\frac{V^2}{2g}$ feet.

Ex. 547.—If a body is thrown upward with a velocity of 200ft. per second find its greatest height. *Ans.* 625 ft.

Ex. 548.—If a body falls freely through 150ft. find the velocity it acquires. *Ans.* 98.

Ex. 549.—A body is projected vertically upward with a velocity of 200ft. per second; how long will it take to reach the top of a tower 200 feet high, and with what velocity will it reach that point?

Ans. (1) 1·1 sec. (2) 164·9.

Ex. 550.—A stone (A) is let fall from a certain point; one second after another stone (B) is let fall from a point 100 ft. lower down; in how many seconds will A overtake B, and what space will it have described?

Ans. (1) $3\frac{5}{8}$ sec. (2) $210\frac{1}{4}$ ft.

Ex. 551.—A stone (A) is let fall from the top of a tower 350ft. high; at the same instant a second stone (B) is let fall from a window 50ft. below the top; how long before A will B strike the ground? *Ans.* 0·35 sec.

Ex. 552.—A stone (A) is projected vertically upward with a velocity of 96 feet per second; after 4 seconds another stone (B) is let fall from the same point; how long will B move before it is overtaken by A, and at what point will this take place?

Ans. (1) 4 sec. (2) 256 ft. below the point of projection.

Ex. 553.—In the last Example if only 3 seconds had elapsed when B was let fall would A ever have overtaken it? *Ans.* No.

107. *Velocity due to a certain height.*—When a body is moving with a given velocity (V), a certain height H can always be found such that if a body fell down it freely from rest it would acquire the given velocity: under these circumstances V is said to be the velocity due to the height H. These quantities are, of course, connected by the equation

$$V^2 = 2gH.$$

Ex. 554.—Determine the height to which velocities of 20, 59, and 760 feet per second are respectively due.

Ans. (1) $6\frac{1}{4}$ft. (2) $54\frac{25}{64}$ft. (3) 9025 ft.

108. *Other cases of uniformly accelerated motion.*—The formulæ hitherto used are true for any value of g, and indeed for the motion of any body whch is acted on in

its line of motion by a force that increases its velocity by equal amounts in equal intervals.

Ex. 555.—At the distance of the moon the accelerating force of gravity is reduced to about $\frac{1}{112}$: if a body fell freely under the action of this force for one hour with what velocity per minute would it then be falling: and in how many seconds would a body falling in the neighbourhood of the earth's surface acquire the same velocity?

***Ans.* (1) $1928\frac{4}{7}$. (2) 1 sec. very nearly.**

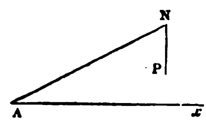
Ex. 556.—If a body were to begin to fall to the earth from the distance of the moon; how many yards would it fall through in half an hour?

***Ans.* 4821 yards.**

Ex. 557.—In the last Example if a body were thrown upward with a velocity of 4 miles an hour, how long would it take to return to the point of projection? *Ans.* 1314 sec.

109. *The motion of a body thrown obliquely in vacuo.* When a body is thrown in a direction making a given angle with the horizon, its position after t seconds is easily determined by the following construction, which, it must be remembered, pre-supposes that the resistance of the air can be neglected.* Let A x be a horizontal line, A N the line of projection; let V be the velocity of projection, set off A N equal to V t on scale; draw N P vertical, and on the same scale make N P equal to $\frac{1}{2} g t^2$; then P will be the position occupied by the body at the end of t seconds. It will be remarked that if the body had been uninfluenced by any external force, it would have reached N at the end of t seconds, and had it been dropped from N it would have reached P in t seconds.

Fig. 134.

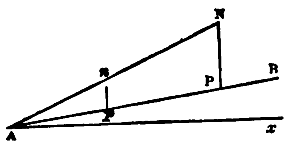


The above rule suggests a very simple construction for the determination of the range and time of flight of a projectile on any plane. Suppose a body to be thrown

* This resistance causes a wide departure from the construction given in the text in all cases in which the velocity is great—*e. g.* in the case of military projectiles. See Appendix II.

from A in a direction of AN with a velocity V, it is required to determine the point (P) at which it strikes AB, and the time (t) which elapses between its leaving A and striking P. From any point (n) in AN draw the vertical line np cutting AB in p; measure the lengths An, np, pA on any scale; now if the point P were known we should have $\mathrm{AN} = \mathrm{V}t$ and $\mathrm{NP} = \frac{1}{2}g\,t^2$; therefore, by similar triangles,

Fig. 135.



$$\mathrm{A}n : np :: \mathrm{V}t : \tfrac{1}{2}gt^2 :: \frac{2\mathrm{V}}{g} : t,$$

a proportion which gives t. Whence AN is known, and then AP can be found by the proportion,

$$\mathrm{A}n : \mathrm{A}p :: \mathrm{AN} : \mathrm{AP}.$$

This construction is true for *any* inclination of the plane, *i.e.* whether it be horizontal or inclined either upward or downward.

Ex. 558.—A body is projected with a velocity of 100 ft. per second in a direction making an angle of 37° with the horizon: determine the time of flight and range on a horizontal plane. *Ans.* 3·76 sec. and 300·4 ft.

Ex. 559.—Determine the time of flight and range on a plane inclined at an angle of 10° upward from the horizon in the case of a body projected as in the last Example. *Ans.* 2·88 sec. and 233·6 ft.

Ex. 560.—A body is projected with a velocity of 120 ft. per second in a direction making an angle of 28° 45′ with the horizon, determine the time of flight and range on a plane passing through the point of projection, (1) when it is horizontal; (2) when inclined upward from the horizon at angle of 12°; (3) when inclined downward at the same angle.

Ans. (1) 3·61 sec. and 379·5 ft. (2) 2·21 sec. and 237·7 ft. (3) 5 sec. and 538·3 ft.

Ex. 561.—A body is thrown horizontally with a velocity of 50 ft. per second from the top of a tower 100 ft. high; find after how long it will strike the ground, and at what distance from the foot of the tower?

Ans. (1) 2·5 sec. (2) 125 ft.

Ex. 562.—If any number of bodies are thrown horizontally from the top of a tower they will all strike the ground at the same instant whatever be the velocities of projection.

110. *The acceleration of the motion of a given body produced by a given pressure.*—Let the weight of the body be W lbs; we have seen that if it falls freely it acquires in every second an additional velocity g. In other words, if this body is acted on by a pressure of W lbs., its velocity is increased every second by g. Now suppose it to be acted on by a constant pressure P; for instance, suppose it to be placed on a smooth horizontal plane and to be pushed by a horizontal pressure of P lbs., it appears from experiment * that its velocity will be increased in every second by a certain constant amount f, given by the proportion

$$W : P :: g : f,$$

that is to say, *the accelerations,* or *the increments of velocity of the same body, in each second are proportional to the pressures that produce them.*

It follows from the remark already made (Art. 108) that the formulæ previously given for falling bodies will be true in the present case when f has been substituted for g. Thus we shall have

$$v = ft \qquad s = \tfrac{1}{2}ft^2 \qquad v^2 = 2fs \quad \&c.$$

Ex. 563.—A body weighing 30 lbs. slides along a smooth horizontal plane under a constant pressure of 15 lbs., determine (1) the additional velocity it acquires in every second; (2) the velocity it will have at the end of 5 seconds; (3) the space it will pass over in 5 seconds.

Ans. (1) 16. (2) 80. (3) 200 ft.

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* It may be remarked that it is very difficult to devise experiments which shall exhibit the fundamental principles of Dynamics in a state of isolation; Galileo, who discovered most of them, possessed a rare sagacity in detecting the *parts* of a phenomenon which were due to disturbing causes, and thus was enabled to get at the fundamental principles. The experimental verification of these principles is nearly always *indirect,* and consists in comparing actual cases of motion (*e.g.* that of planets, of pendulums, &c.) with the secondary principles which have been derived from them.

Ex. 564.—A mass weighing W lbs. is urged along a rough horizontal plane by a pressure of P lbs. acting in a direction parallel to the plane; the coefficient of friction is μ; if the body's velocity is increased in every second by f, show that

$$f = \frac{P - \mu W}{W}\ g.$$

Ex. 565.—A weight of 100lbs. is moved along a horizontal plane by a constant pressure of 20lbs.; the coefficient of friction is 0·17; determine (1) the space it will describe in 10 seconds; (2) the time in which it will describe 200 ft. *Ans.* (1) 48 ft. (2) 20·4 sec.

Ex. 566.—A train weighing 50 tons is impelled along a horizontal road by a constant pressure of 550 lbs.; the friction is 8 lbs. per ton: what velocity will it have after moving from rest for 10 minutes, and what space will it describe in that time?* *Ans.* (1) $17\frac{1}{2}$ miles per hour. (2) 7714 ft.

Ex. 567.—If in the last Example the steam were cut off at the end of the 10 minutes, how many seconds will elapse before the train stops, and how far will it go? *Ans.* (1) 225 sec. (2) 2893 ft.

Ex. 568.—A train is observed to move at the rate of 30 miles per hour, and to run on a horizontal plane for 10,000 ft., find how many lbs. per ton the resistances amount to supposing them independent of the velocity.

[It is easily shown that $f = 0{\cdot}0968$; then the resistance (P) in lbs. per ton (W) is found to equal 6·776 lbs.]

Ex. 569.—A weight Q is tied to a string and rests on a rough horizontal table; to the other end of the string is tied a weight P which hangs vertically over the edge of the table; if the weight of the string and its friction against the edge of the table are neglected, show that when P falls it accelerates Q's velocity in every second by f, where

$$f = \frac{P - \mu Q}{P + Q} \cdot g.$$

[The student will remark that in this case a weight P + Q is moved by a pressure $P - \mu Q$.]

Ex. 570.—A mass of cast iron weighing 100 lbs. is drawn along a horizontal plane of cast iron by means of a cord which is parallel to the plane, and to the end of which a weight of 20 lbs. is attached: determine (1) the acceleration; (2) how far it will move in 4 seconds.

Ans. $1\frac{1}{15}$ ft. per sec. in each second. (2) $8\frac{8}{15}$ ft.

Ex. 571.—If in the last Example the mass had described 5 ft. in $1\frac{1}{2}$ seconds what must have been the coefficient of friction? *Ans.* $\frac{1}{30}$.

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* If the resistances which oppose the motion of a train were constant, it would be possible to attain any velocity however great: in reality the resistance of the air always imposes a limit on the velocity that can be attained by a train moving under a pressure that exceeds the frictions by any given amount: thus Mr. Scott Russell's formula for the resistance contains a term involving the square of the velocity of the train. (Rankine, p. 620.)

Ex. 572.—If in Ex. 569 Q weighs 1 lb. and P weighs 1 oz.; if moreover the length of the string is 12 ft. and P is placed at the edge of the table which is 3 ft. above the ground, find (1) how long P will take to reach the ground; (2) how long it will take Q to arrive at the edge of the table; the friction between Q and the table being neglected.

Ans. (1) 1·78 sec. (2) 4·46 sec.

Ex. 573.—In the last Example suppose P and Q each to weigh one pound, determine the coefficient of friction between Q and the table if that body just reaches the edge. *Ans.* $\frac{1}{7}$.

Ex. 574.—If P and Q are two weights connected by a fine thread (whose weight is neglected) passing over a fixed smooth cylinder; determine the acceleration; and if a weight equal to P − Q is taken from P after it has described s feet, determine the space it will describe in the next t seconds.

Ex. 575.—In the last Example show that the tension on the string before P − Q is removed equals $\frac{2PQ}{P+Q}$.

[Let T be the required tension; the pressure acting on P is P − T *downward*, so that the acceleration of P *downward* is $\frac{P-T}{P}.g$. Similarly the acceleration of Q upward is $\frac{T-Q}{Q}.g$. And these must be equal, since at any instant P is moving downward with the same velocity that Q has upward.]

Ex. 576.—In Ex. 569 show that the tension equals $\frac{(1+\mu)PQ}{P+Q}$.

Ex. 577.—In Ex. 570 find the tension on the string. *Ans.* $19\frac{1}{3}$ lbs.

Ex. 578.—A sphere lies on the deck of a steamer and is observed to roll back 20 inches; if the resistance to rolling is the $\frac{1}{20}$th part of its weight, determine the change in the velocity of the steamer. *Ans.* 2·309 ft. per sec.

111. *The work accumulated in a moving body.*—If a body weighing W lbs. is at any instant moving with a velocity of V feet per second, there will be accumulated in it a certain number of units of work; this is evident from the fact that the moving body is capable of overcoming any given resistance through a certain space; the precise number of units of work thus accumulated is given by the formula $\frac{W}{2g}V^2$; this can readily be proved as follows:—It is plain that the number accumulated at any instant is independent of the direction of the velocity; we may therefore suppose it to be in any direction that will

enable us to ascertain the number; now, if we suppose the body to be moving vertically upward, it will ascend to a height H, given by the formula

$$V^2 = 2gH.$$

But to lift a weight W through H feet requires the expenditure of WH units of work, therefore WH or $\frac{W}{2g}V^2$ is the number of units of work that must have been accumulated in the body.

Ex. 579.—A body whose weight is 10 lbs. moves with a velocity of 16 ft. per second, it has to overcome a constant resistance of half a pound; determine the number of feet it will describe before stopping.

[There are 40 units of work accumulated in the body: now, if x be the number of feet required, $\frac{1}{2}x$ is the number of units of work done; whence x equals 80 feet.]

Ex. 580.—In a similar manner obtain the answers to the Examples 567, 568, 573, and 578.

Ex. 581.—A railway truck weighing with its contents 10 tons—resistances being 8 lbs. per ton—is drawn from rest by a horse; after going 300 ft. it is observed to be moving at the rate of 5 ft. per second: determine the number of units of work that has been done by the horse. *Ans.* 32750.

Ex. 582.—A train weighs one hundred tons—resistances are 8 lbs. per ton—determine the smallest number of units of work expended in a run of 100 miles on a level road.* *Ans.* 422400000.

Ex. 583.—In the last Example if the train stops 10 times and the driver in each case gets the speed up to 30 miles an hour, and to save time turns off the steam and puts on the break at 1000 ft. before each station, determine the total loss of work; and the proportion it bears to the total number of units that need be expended. *Ans.* (1) 59760000. (2) nearly $\frac{1}{7}$th.

Ex. 584.—A shot weighing 6 lbs. leaves the mouth of a gun with a velocity of 1000 ft. per second: determine the number of units of work accumulated in it, and the mean pressure exerted by the exploded powder behind it if the length of the bore is 5 ft. *Ans.* (1) 93750. (2) 18750 lbs.

Ex. 585.—If the shot in the last Example penetrates 24 in. into a piece of sound oak, determine the mean resistance offered by the wood.

Ans. 46875 lbs.

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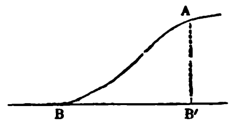
* It is supposed that at the end of the journey the steam is turned off at such a point that the train just runs into the station without putting on the break.

112. *Velocity acquired by a body in sliding down a smooth curve.*—Let h be the *vertical* height of a point A above another point B, the points being anywhere situated; let us suppose them joined by any *smooth* line, whether *straight* or *curved;* then if a body is supposed to slide in vacuo from A to B along the curve, and if V is the velocity it has at A, and v the velocity it has at B, it can be proved that

Fig. 36.



$$v^2 = V^2 + 2gh.$$

Now if a point B′ were to be taken vertically under A, and in the same horizontal line as B, a body that is thrown down from A with a velocity V will have at B′ the same velocity v. In explanation of this remarkable fact it may be observed that at every instant the reaction is perpendicular to the direction in which the body is moving, and therefore cannot accelerate the velocity. The same formula is true of a body suspended by thread, and oscillating; for the tension of the string will act at each point perpendicularly to the direction of the body's motion, and will neither accelerate nor retard its velocity.

Ex. 586.—A stone is tied to the end of a string 10 ft. long and describes a vertical circle of which the string is the radius; if at the highest point it is moving at the rate of 25 ft. per second, find its velocity after describing angles of 90°, 180°, and 270° respectively from the highest point.

Ans. (1) 35·6. (2) 43·6. (3) 35·6.

Ex. 587.—Show that if a body oscillates in any arc of a circle the length of the arc of ascent would always equal the arc of descent if there were no passive resistances.

Ex. 588.—A body is tied to the end of a string 12 feet long, the other end of which is fastened to a point A; at a distance of 4 ft. vertically below A is a peg B; the body descends through an angle of 30° when the string comes to the peg B; find the angle through which the body will rise. *Ans.* 36° 58′.

Ex. 589.—Suppose a body to move in a circle whose radius is r, and lowest point A; let V be the velocity it has at a point P and v that which it has at Q; let the chords AP and AQ or denoted by C and c respectively show that

$$v^2 = V^2 + \frac{g}{r}(C^2 - c^2).$$

113. *Centrifugal force.*—If a stone is tied to the end of a string and whirled round, there arises a very peculiar case of the action of forces, and one which requires careful consideration. Suppose the string to be r feet long, the stone to weigh W lbs. and to move with a velocity of V feet per second; now the tendency of the stone at each instant is to move off in the direction of a tangent to the circle it describes, therefore there must be exerted on it at each instant a certain pressure P (acting along the radius and towards the centre) sufficient to deflect it from the tangent and to keep it in the circle; this pressure is given by the formula

$$P = \frac{W}{g} \cdot \frac{V^2}{r}.$$

In the case supposed this pressure is supplied by the hand, and gives rise to the same sensation as would be produced if the stone were at rest and pulled outward with a pressure of P lbs. It must be added, that when any heavy body moves in a circle under the action of any forces whatever, the sum of the resolved parts of the forces along the radius must at each instant equal $\frac{W}{g} \cdot \frac{V^2}{r}$ or the body will not continue to move in the circle.

We have already seen that if a body whose weight is W is acted on by a pressure P, it would acquire at the end of every second an additional velocity f equal to $\frac{P}{W}g$; in the present case therefore

$$f = \frac{V^2}{r}.$$

The acceleration f is frequently spoken of as the "centrifugal force."

Ex. 590.—A weight of 1 lb. is fastened to the end of a string 3 ft. long and made to perform 50 revolutions in 1 min. with a uniform velocity: the revolutions take place in a horizontal plane, determine the tension on the string. *Ans.* 2·57 lbs.

Ex. 591.—In Ex. 586 determine the tension on the string at the highest and at the other points, supposing the body to weigh 10lbs.

Ans. (1) 9·53lbs. (2) 39·53lbs. (3) 69·53lbs. (4) 39·53lbs.

Ex. 592.—If a body moves in a vertical circle the radius of which is 5ft.; determine the velocity at the highest point that the body may just keep in the circle. *Ans.* 12·65.

[Let T be the tension on the string, then $T + W = \frac{W}{g} \cdot \frac{V^2}{r}$ and the body will just keep in the circle if $T = 0$. If $\frac{W}{g} \cdot \frac{V^2}{r}$ were less than W the body would fall within the circle; if it were greater than W there would be a certain tension on the string.]

Ex. 593.—In the last Example show that the tension on the string at the lowest point will equal 6 times the weight of the body; and that when the body has described a quadrant from the highest point the tension is 3 times the weight of the body.

Ex. 594.—Show that the centrifugal force at the equator equals 0·11129 or the $\frac{1}{289}$th part of what the accelerating force of gravity would be if the earth were at rest.

[See Ex. 528 and Table XV.]

Ex. 595.—Given that the moon makes one revolution round the earth in about 2,360,000 seconds, and nearly in a circle whose radius is 59·964 times the earth's equatorial radius, show that the accelerating force of gravity on the moon must equal $\frac{1}{112·48}$, reckoning in feet and seconds: what inference can be deduced from this as to the law of the decrease of the earth's attraction?

Ex. 596.—A cast iron wheel whose internal radius is 6 feet revolves round a vertical axis, a piece of cast iron is placed without any support on its interior circumference; what number of revolutions must the wheel make per minute in order to keep the iron from falling? *Ans.* 55·13.

[The pressure of the wheel against the iron being P there will be a friction μP which must equal W, the weight of the iron.]

114. *Time of oscillation of a simple pendulum.*—If a small bullet is suspended by a very fine thread, and caused to oscillate in any small arc (*e.g.* not exceeding 2° or 3° on each side of the lowest point), then the time of each oscillation* is given by the formula

$$t = \pi\sqrt{\frac{l}{g}}$$

* *i.e.* the time of moving from the highest point on one side to the highest on the other.

where t is the required time in seconds, and l the distance in feet from the point of suspension to the centre of the bullet. It may be remarked that the above formula would be rigorously true if the bullet were reduced to a point, the thread perfectly flexible, and without weight, and the arc of vibration indefinitely small: a pendulum possessing these properties (which is, of course, an abstraction) is called a *simple pendulum*, and the above formula is said to give the time of a small oscillation of a simple pendulum; in all other cases the above formula gives only an approximation to the time of an oscillation; in the case supposed the approximation is very close.

Ex. 597.—If $g=32\cdot2$ determine the number of oscillations made in one hour by a pendulum 3 ft. long. *Ans.* 3754·2.

Ex. 598.—It is found that at a certain place a pendulum 39·138 inches long oscillates in one second; determine the accelerating force of gravity at that place. *Ans.* 32·1897.

Ex. 599.—If L is the length of a seconds pendulum show that

$$g=\pi^2 L.$$

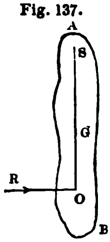
Ex. 600.—If L is the length of a seconds pendulum at any place, and l the length of a pendulum that oscillates in n seconds at the same place, show that

$$l=n^2 L.$$

Ex. 601.—A pendulum at the average temperature oscillates in one second; it is found that its length is L; after a certain time it is found to lose 50 seconds a day; determine the increase of its length. *Ans.* 0·00126 L.

115. *Centre of oscillation and percussion.*—Let AB represent any body capable of oscillating about an axis passing through S perpendicularly to the plane of the paper, which also contains the centre of gravity G: let the body be made to oscillate round the axis, and let the time of its small oscillations be noted; determine the length l of the simple pendulum which would make a small oscillation in the same time; in SG produced take O, so that SO equals l; then the point O is called the *centre of oscillation* of the body corresponding to the *centre of suspension* S. If AB has

Fig. 137.



a definite geometrical form, SO can be determined by calculation, as will be explained hereafter; but in any case it can be determined as above.

In the plane of the paper draw OR at right angles to SO; it admits of proof that if AB* were struck a blow of any magnitude along the line OR, there would be no impulse communicated to the axis; the point O is therefore also called the *centre of percussion.*

Ex. 602.—A mass of oak is suspended freely by a horizontal axis; it is observed to make 43 oscillations in one minute; at what distance below the point of support must a shot be fired into it so that there may be no impulsive strain on the point of support? *Ans.* 6·313 ft.

Ex. 603.—A tilt hammer when allowed to oscillate about its axis is observed to make 35 small oscillations per minute, at what distance from its axis must be the point at which it strikes the object on the anvil in order that no impulse may be communicated to the axis? *Ans.* 9·528 ft.

116. *Variations in the accelerating force of gravity at different places of the earth's surface.*—When experimental determinations of the accelerating force of gravity are made with great care, it is found to have different values at different places; these variations are due to two principal causes. (1.) The spheroidal form of the earth, in consequence of which the attraction of the earth at different places is not the same. (2.) The diurnal rotation of the earth, which causes the effective or apparent force of gravity to be less than the actual attraction, as part of the latter is consumed in keeping bodies on the surface. Besides these general causes, variations are produced in the determinations made at particular places by differences in their level, and differences in the density of the strata in their immediate neighbourhood. The effective force of gravity at any place is determined by ascertaining the length L of a simple pendulum which beats seconds at

* The body AB is supposed to be symmetrical with reference to the plane of the paper.

that place, and then the accelerating force of gravity is determined by the formula (Ex. 599)

$$g = \pi^2 L.$$

The following table gives the lengths of the seconds pendulum at different places, according to Mr. Airy*, and the values of g which can be deduced from them.

TABLE XV.

THE VALUE OF THE ACCELERATING FORCE OF GRAVITY AT DIFFERENT PLACES.

Observer.	Place.	Latitude.	Length of seconds pendulum in inches.	Accelerating force of gravity; feet and seconds.
Sabine . . .	Spitzbergen .	N. 79° 50′	39·21469	32·2528
Sabine . . .	Hammerfest .	70° 40′	39·19475	32·2363
Svanberg . .	Stockholm . .	59° 21′	39·16541	32·2122
Bessel . . .	Königsberg . .	54° 42′	39·15072	32·2002
Sabine . . .	Greenwich . .	51° 29′	39·13983	32·1912
Borda, Biot and Sabine . . .	Paris	48° 50′	39·12851	32·1819
Biot	Bordeaux . .	44° 50′	39·11296	32·1691
Sabine . . .	New York . .	40° 43′	39·10120	32·1594
Freycinet . .	Sandwich Islnds	20° 52′	39·04690	32·1148
Sabine . . .	Trinadad . .	10° 39′	39·01888	32·0913
Freycinet . .	Rawak . . .	S. 0° 2′	39·01433	32·0880
Sabine and Duperrey . . .	Ascension . .	7° 55′	39·02363	32·0956
Freycinet and Duperrey . .	Isle of France .	20° 10′	39·04684	32·1151
Brisbane and Rumker . .	Paramatta . .	33° 49′	39·07452	32·1375
Freycinet and Duperrey . .	Isles Malouines	51° 35′	39·13781	32·1895

* Figure of the Earth, p. 229.

CHAP. II.

ON ACCELERATED MOTION.

117. *Accelerating force.*—If the velocity of a body is continually increased by equal amounts in equal times, that velocity is said to be *uniformly accelerated;* and the cause which produces this acceleration is said to be a *uniformly accelerating force.*

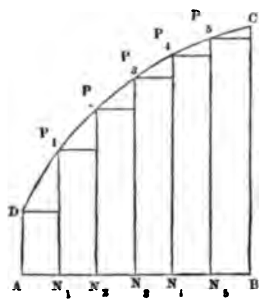
Obs.—If the velocity of a body is continually diminished by equal amounts in equal times, it is said to be *uniformly retarded;* and the cause which produces this effect is said to be a *uniformly retarding force.* Now, it must be remarked, the same cause may produce uniform acceleration in one body, and uniform retardation in another: thus gravity produces the former effect on a body moving downward, and the latter on a body moving upward: for this reason the term "retarding force" is rarely used, and the *accelerating force* of gravity is spoken of, whether the body is moving upward or downward. It may be remarked, however, that some forces, such as friction, are essentially retarding forces. It must be carefully borne in mind that a body is said to be acted on by a uniformly accelerating force f, when at the end of each second its velocity is increased by a velocity of f feet per second.

Proposition 22.

If ABCD be any area bounded by a line straight or curved CD, and by straight lines AB, AD, BC, of which

the two latter are at right angles to AB; and if the line DC be such that for any point P the abscissa PN represents the velocity with which a body moves at the end of a time t, that is represented on the same scale by AN, then the area of the curve will represent the space described in the time AB.

Fig. 128.



Divide AB into any number of equal parts in N_1, N_2, N_3 .... draw the ordinates P_1N_1, P_2N_2, P_3N_3, ... and complete the rectangles DN_1, P_1N_2, P_2N_3, ... Now, if we suppose the body to move during each interval of time with the velocity it has at the commencement of that interval, then (Art. 103) it will describe a space represented by the sum of the rectangular areas DN_1, P_1N_2, P_2N_3,....; and this will be true, however great the number of intervals may be, and therefore when the velocity changes continuously, the space described will be correctly represented by the limit of the sum of those areas, *i.e.* by the curvilinear area ABCD.

Proposition 23.

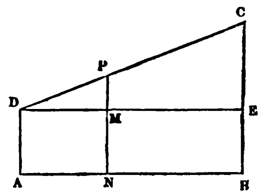
If a body begins to move with a velocity of V feet per second, and is acted on by a uniformly accelerating force f along the line of its motion, the number of feet (s) described by it in t seconds is given by the formula

$$s = Vt + \tfrac{1}{2}ft^2.$$

Let AB represent the time t on scale; at right angles to AB draw AD and BC representing on the same scale V—

the velocity at the beginning of the motion—and $V + ft$—the velocity at the end of the motion (Art. 104); join DC, then the area ABCD will represent the space s.

Fig. 139.



Draw DE parallel to AB; in DC take any point P, and draw PN parallel to DA, cutting DE in M. Now, since DA (which represents V) is equal to BE, we shall have CE equal to ft, or $f \times AB$. But by similar triangles

$$PM : CE :: DM : DE :: AN : AB$$

and CE equals $f \times AB$; therefore PM equals $f \times AN$, therefore PN equals $AD + f \times AN$, *i.e.* (Art. 104) represents the velocity with which the body is moving at the end of the time represented by AN; hence the area ABCD represents the required space s (Prop. 22).

$$\text{Now area } ABCD = \tfrac{1}{2} AB\,(AD + BC)$$
$$\therefore \quad s = \tfrac{1}{2} t\,(V + V + ft)$$
$$\text{or} \quad s = Vt + \tfrac{1}{2} ft^2.$$

Cor.—If the body begins to move in a direction opposite to that in which the force acts—*i.e.* if its velocity is uniformly retarded—a precisely similar process leads to the equation

$$s = Vt - \tfrac{1}{2} ft^2$$

and if the body begins to move from rest, we shall obtain

$$s = \tfrac{1}{2} ft^2.$$

Ex. 604.—If a body is thrown up with any velocity and if t_1 and t_2 are the times during which it is respectively above and below the middle point of its path, show that

$$t_1 : t_2 :: 1 : \sqrt{2} - 1.$$

Ex. 605.—If a body falls under the action of any uniformly accelerating force f, show that the spaces described in successive seconds form an arithmetical series of which the first term is $\frac{f}{2}$ and the common difference f.

Ex. 606.—If a body is let fall and describes a certain space, and this space is divided into n equal parts, show that the time of describing the first part is to that of describing the last as 1 is to $\sqrt{n} - \sqrt{n-1}$.

Ex. 607.—There is a chasm with water at the bottom, on dropping a stone down it the splash is heard n seconds after the stone leaves the hand; show that the distance of the surface of the water below the hand is given by the formula $(g = 32{\cdot}2)$

$$s = 1130\,(35 + n - \sqrt{1225 + 70\,n}).$$

[The velocity of sound may be taken at 1130 feet per second; it will be observed also that $1130 \div 16{\cdot}1 = 70$ very nearly; now let x be the time the stone takes to fall, $n - x$ is the time the sound takes to rise, and if s is the required depth we have

$$s = \tfrac{1}{2}\,g x^2 = 16{\cdot}1\;x^2$$
$$\text{and } s = 1130\,(n - x)$$
$$\therefore\; x^2 = 70\,(n - x)$$

whence s is easily found.]

Ex. 608.—When n is but a few seconds, show that the formula in the last Example can be written

$$s = \frac{565\,n^2}{35 + n}.$$

Ex. 609.—Determine the values of s from the formulæ of Examples 607 and 608 when n equals 3, 4 and 5 seconds respectively.

Ans. (1) 134·3 ft. and 133·8 ft. (2) 232·4 ft. and 231·8 ft. (3) 354·5 ft. and 353·1 ft.

Ex. 610.—If V is the velocity of sound, show that the formulæ in Examples 607 and 608 become respectively

$$s = \mathrm{V}\left\{\frac{\mathrm{V}}{g} + n - \sqrt{\left(\frac{\mathrm{V}}{g}\right)^2 + \frac{2\mathrm{V}n}{g}}\right\} \text{ and } s = \frac{\mathrm{V}}{2}\cdot\frac{n^2}{\frac{\mathrm{V}}{g} + n}.$$

and hence show that if an error of k* feet is committed in the numerical value assigned to V, the error in the value of s is very nearly

$$\frac{kg^2\,n^3}{2\,(\mathrm{V} + g\,n)^2}.$$

Ex. 611.—What is the error produced in the determination of s by an error of 20 ft. in the velocity of sound (assumed as 1130) when n equals 5 seconds? *Ans.* 0·78 ft.

118. *Change in the numerical value of an accelerating force produced by a change in the units of space and time.*—We have hitherto taken the numerical value of the

* It must be remembered that the velocity of sound is different in different states of the atmosphere.

accelerating force of gravity to be 32, which presupposes that space is measured in feet, and time in seconds; the use of these units is of course arbitrary, and we might employ any others; the question then arises, were this done what numerical value must be assigned to the accelerating force of gravity? The method of obtaining this value will be readily understood by considering the following Examples:—

Ex. 612.—What velocity would be acquired by a body that fell freely for one minute? *Ans.* 1920 ft. per sec.

Ex. 613.—If a body moves with a velocity of 1920 ft. per second, what is its velocity estimated in yards per minute? *Ans.* 38400.

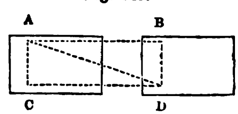
[Now it must be remembered that at the end of each *second* a falling body acquires an additional velocity of 32 *feet* per *second;* it appears from the last two Examples that the same body would acquire in each *minute* an additional velocity of 38400 *yards* per *minute;* but the former number represents the accelerating force of gravity in feet and seconds, and therefore the latter represents the accelerating force of gravity in yards and minutes.]

Ex. 614.—Given that the accelerating force of gravity at the distance of the moon equals $\frac{1}{112}$ in feet and seconds, find its value in miles and hours. *Ans.* 21·91.

Ex. 615.—An accelerating force has a numerical value f when certain units are employed, show that its value will be $\frac{m^2 f}{n}$ when the new unit of time contains m, and the new unit of space n, of the old units respectively.

119. *Composition of velocities.*—Suppose a body to be at any instant at the point A, and suppose it to be moving with such a velocity as would in a certain time carry it to B along the line AB; suppose that at that instant another velocity were communicated to it such as would in the same time carry the body along the line AC to C, if it had moved with that velocity only; complete the parallelogram ABCD, and join AD, then at the end of the given time the body will arrive at D, having moved along the line AD. That this is so appears at once from the well known fact, that when a ship is in a state of steady motion, a

Fig. 140.



man can walk across her deck with precisely the same facility as if she were at rest; thus if he were to walk across the deck when the ship is at rest he would go from A to C; but if we suppose the ship to have such a velocity as will in the same time carry the point A to B, he will come to the point D; and if the velocities have been uniform he will have moved along the line AD. Now, let v and u be the two velocities, then

$$AB : AC :: u : v$$

and if V is the velocity compounded of them

$$AD : AB :: V : u.$$

So that if AB and AC represent the given velocities in magnitude and direction, AD will represent the velocity compounded of them in magnitude and direction. Hence the rule for the composition of velocities is the same, *mutatis mutandis*, as that for the composition of pressures.

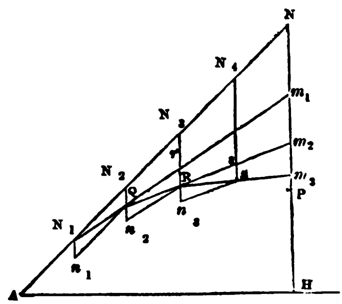
If the velocities vary from instant to instant, the rule will give the magnitude and direction of the component velocity at any instant; this is the case which commonly happens, for example, when a body is thrown in any direction transverse to the action of gravity; the method of determining the motion of the body may be described in general terms as follows:—Conceive the time to be divided into a great number of intervals, and suppose the velocity that is actually communicated by gravity during each interval, to be communicated at once*, then, by the composition of velocities, we can determine the motion during each interval, and therefore during the whole time; the actual motion is the limit to which the motion, thus determined, approaches when the number of intervals is increased.

* It is immaterial whether we conceive it to be communicated at the beginning or at the end of the interval.

Proposition 24.

A body is thrown in vacuo in a direction making any angle with the horizon, and with a given velocity (V) to determine its position at the end of any given time (t).

Fig. 141.



Let the body be projected along the line AN; take AN equal to Vt, and divide t into n parts, each equal to τ; then if AN is divided into the same number of equal parts in $N_1, N_2, N_3 \ldots$ each part will equal $V\tau$. Now, the effect of gravity is to increase the velocity of a falling body by $g\tau$ in a time τ; we may therefore conceive the body to move through each interval with the velocity it has at the beginning of that interval, and at the end the velocity to be compounded with $g\tau$. During the first interval the body will move over the space AN_1; draw $N_1 n_1$ vertical and equal to $g\tau \times \tau$; complete the parallelogram $n_1 N_2$; then since the sides $N_1 n_1$ and $N_1 N_2$ are proportional to the velocities $g\tau$ and V, the body will, during the next interval, move along the line $N_1 Q$, and at the end of the interval will arrive at a point Q vertically under N_2; the actual velocity with which the body has moved being, of course, equal to $N_1 Q \div \tau$. At the point Q we have to compound this velocity with $g\tau$; to do this we must produce $N_1 Q$ to r, making Qr equal to $N_1 Q$; take Qn_2 equal to $g\tau \times \tau$, and complete the parallelogram, then the sides of this figure are proportional to the component velocities, and therefore the diagonal is in the same proportion to

the velocity compounded of them; at the end of the third interval therefore the body will be at R vertically under N_3; the same construction will apply to any number of intervals, and the required point P will be vertically under N. To determine NP; produce N_1Q to m_1, QR to m_2, RS to m_3, &c., then will NP equal the limit of the sum of Nm_1, m_1m_2, m_2m_3, &c.; but by similar triangles Nm_1 is the same multiple of N_2Q that N_1N is of N_1N_2, therefore Nm_1 equals $(n-1)\,g\tau^2$, similarly m_1m_2 equals $(n-2)\,g\tau^2$, m_2m_3 equals $(n-3)\,g\tau^2$, &c., and therefore their sum equals

$$(n-1)\,g\tau^2+(n-2)\,g\tau^2+(n-3)\,g\tau^2+\ldots+2g\tau^2+g\tau^2$$
$$=g\tau^2\left\{(n-1)+(n-2)+\ldots+2+1\right\}$$
$$=g\frac{t^2}{n^2}\cdot\frac{n(n-1)}{2}=\tfrac{1}{2}gt^2\left(1-\frac{1}{n}\right).$$

Now however great the number of intervals Q, R, S, &c. will remain vertically under N_1, N_2, N_3, &c., so that in the limit P will remain vertically under N. Also the limit of $\frac{1}{2}gt^2\left(1-\frac{1}{n}\right)$ is $\frac{1}{2}gt^2$; so that the true position of the body will be found by measuring downward from N a distance equal to $\frac{1}{2}gt^2$. (See Art. 109.)

Ex. 616.—If a body is projected with a velocity V in a direction making an angle α with the horizon, show that the time of flight is $\frac{2V}{g}\sin\alpha$ and the horizontal range $\frac{V^2\sin 2\alpha}{g}$.

[See Art. 109.]

Ex. 617.—If a body is thrown with a given velocity the horizontal range is greatest when it is projected at an angle of 45°; and for angles of projection one as much less as the other is greater than 45° the horizontal ranges are the same.

Ex. 618.—Show that the least velocity with which a body can be projected to have a horizontal range R is $4\sqrt{2R}$ feet per second.

Ex. 619.—Determine the angle of elevation and velocity of projection that will enable a body to strike the ground after 10 seconds at a distance of 5000 ft. from the point of projection.

Ans. (1) $17° 45'$. (2) 525 ft. per sec.

Ex. 620.—A body is projected with a velocity V in a direction making an angle α with the horizon, if R is its range on a plane passing through the point of projection and inclined at an angle θ to the horizon*, and T the time of flight, show that

$$R = \frac{2V^2}{g} \cdot \frac{\sin(\alpha-\theta)\cos\alpha}{\cos^2\theta} \text{ and } T = \frac{2N}{g} \cdot \frac{\sin(\alpha-\theta)}{\cos\theta}.$$

[See Art. 109.]

Ex. 621.—There is a hill whose inclination to the horizon is 30°; a projectile is thrown from a point on it at an angle inclined to the horizon at 45°; show that if it were projected down the plane its range would be nearly $3\frac{3}{4}$ times what the range would be if it were thrown up the plane.

Ex. 622.—In the last Example suppose the slope of the hill to be due north and south, and the azimuth of the plane of projection to be A; show that the sum of the two ranges obtained by throwing the body towards the ascending and descending parts of the hill equals

$$\frac{2V^2}{g}\sqrt{1+\frac{\cos^2 A}{3}}.$$

[The azimuth is the bearing of a point from the south measured on a *horizontal plane.*]

Ex. 623.—If in Example 620 the body is so projected as to obtain the greatest range with a given velocity, show that the direction of projection must bisect the angle between the vertical and the plane.

[It must be remembered that $2\sin(\alpha-\theta)\cos\alpha = \sin(2\alpha-\theta) - \sin\theta$.]

Ex. 624.—Referring to the figure in Prop. 24, if AH and HP are denoted by x and y respectively show that

$$x = Vt\cos\alpha$$
$$y = Vt\sin\alpha - \tfrac{1}{2}gt^2.$$

Ex. 625.—Show that the highest point the projectile can reach is $\frac{V^2}{2g}\sin^2\alpha$ feet above the middle point of the horizontal range.

Proposition 25.

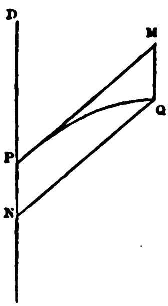
The curve described by a projectile in vacuo is a parabola whose directrix is horizontal, and at a distance above

* The inclined plane is, of course, perpendicular to the plane of projection, unless the contrary is specified.

the point of projection equal to that to which the velocity of projection is due.

Let P be the point and PM the direction of projection; let PQ be the path of the projectile and Q its position at the end of t seconds; draw the vertical lines DPN and MQ, also draw QN parallel to PM, then

Fig. 142.



$$PN = QM = \tfrac{1}{2}\, gt^2$$
$$NP = QM = Vt$$
$$\therefore\; QN^2 = \frac{2V^2}{g} . PN.$$

Now this relation between QN and PN is the same wherever on the curve we may take Q; but if a parabola were drawn through P touching PM, with its diameter vertical and its directrix passing through a point D so taken that 4 PD equals $\frac{2V^2}{g}$ we should have for any point of it

$$QN^2 = \frac{2V^2}{g} . PN$$

i. e. it would coincide with the curve described by the projectile: hence that curve is a parabola whose directrix is horizontal and passing through the point D; but it will be remarked that

$$V^2 = 2g . DP.$$

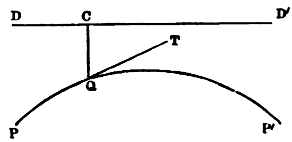
So that DP is the height to which the velocity of projection is due. (See Art. 107.)

Cor.—The velocity of the projectile at any point is that due to the height of the directrix above that point.

Let PQP′ be the path of the projectile, of which DD′ is the directrix; at the point Q let the body be moving with a velocity v in the direction QT; now it is plain that if another body were thrown from Q in the direction QT

with an equal velocity v, it would move in exactly the same manner as the projectile, *i.e.* would describe the curve PQ'; but if a body is thrown from Q so as to describe that curve, it must be thrown with the velocity due to the height QC, *i.e.* $v^2 = 2g\,CQ$.

Fig. 143.



Ex. 626.—Show that the velocity v of the projectile at any time t is given by the formula

$$v^2 = V^2 - 2Vgt \sin \alpha + g^2 t^2.$$

Ex. 627.—There is a wall b feet high, a body is thrown from a point a feet on one side of it so as just to clear the wall and to fall c feet on the other side; show that

$$\tan(\text{angle of elevation}) = \frac{b(a+c)}{ac}$$

$$(\text{vel. of projection})^2 = \frac{g(a+c)}{2}\left\{\frac{ac}{b(r+c)} + \frac{b(a+c)}{ac}\right\}.$$

120. *The first and second laws of motion.*—The object of the first law of motion is to assert that a body has no power of changing its own state of rest or motion, and that every such change is due to the action of some external force; up to the time of Galileo it was supposed that certain kinds of motion—such as the rolling of a body along a road—have a natural tendency to decay; while certain other kinds—such as that of falling bodies—have a natural tendency to increase; when this opinion came to be examined, it was found that every case of "decay" could be referred to the action of retarding forces, *e.g.* friction and resistance of the air, and that the "decay" could be made indefinitely slower by diminishing these resistances; on the other hand every case of increased velocity could be referred to the action of an accelerating force such as gravity. The law is stated as follows: "A body not acted on by any external force, if at rest,

will continue at rest, and if in motion will continue to move uniformly on a straight line." The object of the second law of motion is to assert that the effect produced by a force is irrespective of the previous motion of the body: it is enunciated thus:—"When a force acts on a body in motion, the velocity it would produce in the body moving from rest is *compounded* with the previous velocity of the body." If the body is moving along the line of action of the force, the term *compounded* must be understood to mean added (or subtracted); if the body is moving transversely to the line of action of the force, the word compounded must be understood as in Art. 119. The principle asserted in the second law of motion is illustrated by many obvious facts, such as the following:—a person, on board a ship, can throw up a ball and catch it with equal facility whether the ship is at rest or in a state of steady motion.

CHAP. III.

ON MOTION PRODUCED BY PRESSURE.

121. *Acceleration produced by a given pressure.*—The following examples can be solved by means of the principle laid down in Art. 110.

Ex. 628.—If a body slides down a smooth inclined plane, show that the accelerating force equals $g \sin \alpha$, where α is the inclination of the plane to the horizon; if the plane is rough show that the accelerating force equals $g \dfrac{\sin(\alpha - \phi)}{\cos \phi}$ where ϕ is the limiting angle of resistance.

[In the case of a smooth plane the pressure producing motion is the part of its weight resolved along the plane, *i.e.* is $W \sin \alpha$, whence $f = g \sin \alpha$. In the case of the rough plane the pressure producing motion is $W \sin \alpha$ diminished by the friction, *i.e.* $W \sin \alpha - \mu W \cos \alpha$ or $W \dfrac{\sin(\alpha - \phi)}{\cos \phi}$ whence $f = g \dfrac{\sin(\alpha - \phi)}{\cos \phi}$.]

Ex. 629.—Find the velocity acquired by a body in descending a smooth inclined plane 50 feet long and having an inclination of 23°; determine also the velocity that would be acquired if the limiting angle of resistance is 15°. *Ans.* (1) 35·4 ft. per sec. (2) 21·5 ft. per sec.

Ex. 630.—If a body begins to ascend an incline show that the retarding force is $g \dfrac{\sin(\alpha + \phi)}{\cos \phi}$.

Ex. 631.—There is a plane 50 feet long and inclined to the horizon at an angle of 30°; the limiting angle of resistance between it and a given body is 15°; determine the velocity the body must have at the foot of the plane so as just to reach the top, and the *time* it will take to get there.

Ans. (1) 48·4 ft. per sec. (2) 2·07 sec.

Ex. 632.—If a body slides down a gentle incline of 1 foot vertical to m horizontal, show that the accelerating force very nearly equals $\left(\dfrac{1}{m} - \mu\right) g$. And if $m = 100$ show that the error equals about $\frac{1}{20000}$th part of the whole.

Ex. 633.—A train moving at the rate of 24 miles an hour comes to the top of an incline of 1 foot in 350; the resistances are 8 lbs. per ton; the steam is cut off at the top of the incline and the train comes to rest at its foot, determine (1) the retarding force on the train; (2) the length of the incline; (3) the time of motion. *Ans.* (1) $\frac{4}{175}$. (2) 27104 ft. (3) 1540 sec.

Ex. 634.—At the slide at Alpnach the first declivity has an inclination of $22° 30'$ and is 500 feet long; being kept continually wet the limiting angle of resistance is $14°$; in how many seconds would a tree descend this first declivity were it not for the resistance of the air? *Ans.* 14·3 sec.

Ex. 635.—A body slides from rest down a plane whose inclination is I and length L; it passes with the velocity acquired during the descent of the first plane to a second whose inclination i is less than the limiting angle of resistance ϕ; if l is the space through which it slides before coming to rest, show that

$$\frac{l}{\mathrm{L}}=\frac{\sin(\mathrm{I}-\phi)}{\sin(\phi-i)}.$$

Ex. 636.—A body weighs W lbs. and is pulled up an inclined plane by a pressure P that acts parallel to the plane, show that the accelerating force equals $\left(\frac{\mathrm{P}}{\mathrm{W}}-\frac{\sin(\alpha+\phi)}{\cos\phi}\right)g$, where α is the angle of inclination and ϕ the limiting angle of resistance.

Ex. 637.—Let A C, C B be two planes sloping downward in contrary directions from the point C, and inclined to the horizon at angles A and B respectively; a weight P slides down C A and draws a weight Q up C B by means of a fine cord which passes over C and is tied to each weight; if the limiting angle of resistance between the weights and the planes is ϕ show that

$$f=\frac{\mathrm{P}\sin(\mathrm{A}-\phi)-\mathrm{Q}\sin(\mathrm{B}+\phi)}{(\mathrm{P}+\mathrm{Q})\cos\phi}\cdot g.$$

Ex. 638.—In the last case if the inclines are equal and small, being 1 in m, show that

$$f=\left\{\frac{1}{m}\cdot\frac{\mathrm{P}-\mathrm{Q}}{\mathrm{P}+\mathrm{Q}}-\mu\right\}\cdot g.$$

Ex. 639.—If the resistances are 8 lbs. per ton and the incline 1 in 140, and a set of full trucks are required in their descent to pull up the incline an equal number of similar empty trucks, show that the contents of each truck should on the average be more than double the weight of the truck.

Ex. 640.—If a circle is placed with its plane vertical, and through its highest point any chord is drawn, a body will descend along that chord (supposed to be smooth) in the same time as down the vertical diameter.

Ex. 641.—If through any point there is drawn a vertical line and any number of inclined planes on the same side of the line and having a common limiting angle of resistance ϕ; then if bodies begin to slide down these

planes at the same instant, show that after any interval they will be found in the arc of the segment of a vertical circle cut off by the vertical line which subtends at the centre an angle equal to $\pi - 2\phi$.

122. *The work accumulated in a moving body.*—The following examples depend on the principle proved in Art. 111.

Ex. 642.—A train moving at the rate of 15 miles an hour comes to the foot of an incline of 1 in 300, resistances 8 lbs. per ton; how far will it go before stopping? *Ans.* 1095 ft.

[If W is the weight of the train in lbs. the number of units of work accumulated in it is $\frac{W(22)^2}{2g}$; now if l is the horizontal length of the plane the units of work required to draw W over this length is by Example 468 $Wl\left\{\frac{1}{300}+\frac{1}{280}\right\} \therefore l = \frac{22^2 \times 300 \times 280}{580 \times 2g}$ the number of feet required. The same answer can be obtained by the principle exemplified in the last article.]

Ex. 643.—A body slides down an inclined plane the height of which is 12 feet and length of base 20 feet; find how far it will slide along a horizontal plane at the bottom, supposing the coefficient of friction on both planes to be $\frac{1}{6}$, and that it passes from one plane to the other without loss of velocity. *Ans.* 52 ft.

[From Ex. 468 it appears that the body arrives at the bottom of the plane with a number of units of work accumulated in it equal to $W\left\{12-\frac{20}{6}\right\}$.]

Ex. 644.—If the velocity of a moving body changes from V to v, show that the number of units of work accumulated during the change equals $\frac{W}{2g}(v^2 - V^2)$.

Ex. 645.—A train weighing 90 tons comes to the foot of an incline of 1 in 160 with a velocity of 30 miles an hour, the resistances are 7 lbs. per ton, the length of the incline 2 miles; the train has at the top of the incline a velocity of 20 miles an hour; how many units of work have been expended in getting the train up the incline? And through how great a distance would an expenditure of the same number of units have taken the train with a uniform velocity along a horizontal line?

Ans. (1) 16,570,400 units. (2) 26302 ft.

Ex. 646.—If a train begins to descend the incline in the last Example with a velocity of 20 miles an hour, how far will it descend before acquiring a velocity of 30 miles an hour? *Ans.* 5380 ft.

Ex. 647.—There are two points A and B on a railroad 4 miles apart on the same horizontal line, the railroad is in two equal inclines one up and the other down of 1 in 160; the train, which weighs 50 tons and experiences resistances equal to 7 lbs. per ton, has a velocity of 30 miles an hour at A and B, and a velocity of 20 miles an hour at the top of the incline; the velocity being supposed to change uniformly from 30 to 20 and again from 20 to 30, and when the latter velocity is attained further acceleration is checked by putting on the break; determine (1) the loss in units of work in consequence of the incline; (2) the loss of time in consequence of the incline. *Ans.* (1) 1,813,000 units. (2) $72\frac{1}{8}$ sec.

Ex. 648.—A chest 6 feet long and 2 feet square stands on its end on the deck of a ship, one face being perpendicular to the direction of the motion; the ship is suddenly brought to rest, what must be its velocity if the chest is just overthrown; it being supposed that all sliding is prevented?

Ans. 2·2 miles per hour.

[If W is the weight and V the required velocity the number of units of work accumulated in it must be $\frac{W}{2g}V^2$; and to overthrow the chest requires $W(\sqrt{10}-3)$ units of work.]

Ex. 649.—Show from the principles of the present article that the velocity acquired by the bodies in Ex. 637 while moving from rest over a length l of the planes is given by the formula

$$V^2 = 2gl \cdot \frac{P\sin(A-\phi) - Q\sin(B-\phi)}{(P+Q)\cos\phi}.$$

Ex. 650.—There is an inclined plane of 1 in 90 along which a train weighing 80 tons is made to descend for a distance of 300 feet; to the train is attached a rope which, after passing round a pulley at the top of the incline, is fastened by the other end to a lighter train weighing 16 tons and so long that the light train is at the foot of the incline when the heavy one is at the top; find (1) the velocity with which the heavy train reaches the foot of the incline; (2) if the heavy train is disconnected from the light one at the foot of the incline find the distance to which it will run before stopping on the horizontal plane, resistances on the incline being 7 lbs. per ton, on the level 8 lbs. per ton. *Ans.* (1) 9·07 ft. per sec. (2) 359·7 ft.

123. *Mass, momentum, and moving force.*—The term *mass* is of frequent occurrence in dynamics, and it is of great importance that the student should obtain a clear conception of its meaning, and of the distinction that exists between the *mass* of a body and its *weight;* for this purpose let us consider a particular case. Neglecting

variations of temperature, it is plain that two cubic inches of lead contain twice as much lead as one cubic inch, and so on in any proportion; also at the same place the two cubic inches of lead weigh twice as much as one cubic inch of lead; consequently the weight (W) of a piece of lead will vary as the quantity of lead (M) which it contains. Moreover at different places the weight of a piece of lead, *i. e.* its power to compress a given spring, varies as the accelerating force of gravity at that place*; that is to say, if the force of gravity were doubled, the weight of the same piece of lead would be doubled, and so on in any proportion. Hence the weight must vary as the quantity of lead (M), and the accelerating force of gravity (g) jointly, *i. e.* $W \propto Mg$

$$\text{or } W = kMg$$

where k is a constant number depending on the particular units employed. Now, it further appears that whatever be the physical nature of a body, *i.e.* whether it be lead, or iron, or gold, the quantity denoted by M is the same for all dynamical purposes, and as in each particular case it would denote the quantity of lead, or iron, or gold, we shall correctly generalise its denomination if we call it *the quantity of matter* or *the mass* of the body.

Def.—The momentum of a body is the product of its mass and its velocity.

It will be remarked that the momentum of the body is referred to its *mass*, and not to its *weight;* the reason for doing so is this:—a cubic inch of lead moving with a given velocity would strike the same blow whether the accelerating force of gravity were 32, or had any other value,

* The evidence for this fact is, of course, experimental, see note to Art. 110. It may be added that the variations of the *weight* of the same body at different parts of the earth's surface could probably be observed directly and with great accuracy by means of a delicate spring.—*Herschel's Outlines of Astronomy*, Art. 234.

that is to say, its momentum must not be made to depend on the weight, which varies with the force of gravity, but on the mass, which is irrespective of its variations.

Def.—Moving force, or the moving quantity of a force, is the additional momentum which it communicates in each second to a body.

The accelerating force (f), or the acceleration produced by a force, is the additional *velocity* it communicates in each second; consequently if M is the mass of the body,

$$\text{Moving force} = Mf.$$

124. *The third law of motion.*—We have already seen (Art. 110) that when a pressure P acts upon a body which weighs W lbs. at a place where the accelerating force of gravity is g; then

$$P = \frac{W}{g} f$$

$$i.e.\ P \propto Mf.$$

This variation when verbally enunciated becomes what is commonly called the third law of motion, viz.—*when pressure produces motion it is proportional to the moving force.*

Ex. 651.—If we assume that $W = Mg$, where g is taken in feet and seconds, and W in lbs., how many cubic inches of water will be the unit of mass? *Ans.* 892·60.

[See Art. 2. The value of g at London may be taken to equal 32·192.]

Ex. 652.—If a substance contains V cubic inches and its specific gravity is s, show that the numerical value of M is $\frac{Vs}{892 \cdot 60}$.

Ex. 653.—A cubic foot of cast iron is observed to increase its velocity by 3 feet every second, determine from the last Example the pressure that produces this acceleration. *Ans.* 41·86 lbs.

[The reader must remember that since $W = Mg$ we shall have $P = Mf$.]

Ex. 654.—The accelerating force of the moon's attraction on a point situated on its surface is about 5·4.* A man can jump to a height of 5 feet on the earth's surface; how high could he jump on the moon's surface? *Ans.* 29·7 ft.

* Herschel's Outlines of Astronomy, Art. 508.

Ex. 655.—If equal pressures (P) act on two unequal bodies for the same time, show that the bodies will acquire equal momenta.

Ex. 656.—Show that momenta of the bodies in the last Example will be equal when P varies from instant to instant, provided the pressures are the same at the same instant throughout their time of action.

[The results in the last two Examples are of considerable importance; they are almost self-evident and therefore liable to be forgotten—for this reason the student's attention is particularly directed to them.]

Ex. 657.—When the powder in the bore of a cannon is exploded the pressures on the end of the bore and on the shot are at each instant equal: a shot weighing 6 lbs. is fired from a gun quite free to move and weighing 6 cwt.; the velocity with which the shot leaves the gun is 1000 ft. per second, what is the velocity of the gun's recoil? *Ans.* 8·93 ft. per sec.

Ex. 658.—Show that the number of units of work accumulated in the gun is always small compared with the number accumulated in the shot; and ascertain these numbers in the case suggested in the last Example.

Ans. 93,750 units in the shot and 837 in the gun.

Ex. 659.—What reason can be assigned for the practical rule that, *cæteris paribus*, the velocity of the shot is proportional to the square root of the weight of the charge? *

Ex. 660.—If the trunnions of the gun in Ex. 657 are supported on two parallel smooth planes inclined at an angle of 30°, determine how far it will move along these planes. *Ans.* 2·5 ft.

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* Poncelet, Introd. à la Mécan. Indust., Art. 176.

CHAP. IV.

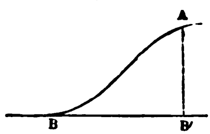
THE CONSTRAINED MOTION OF A POINT.

Proposition 26.

THE velocity acquired by a body in sliding from one point to another on a smooth curve is the same as that acquired by a body which falls freely through a space equal to the vertical height of the higher above the lower point.

Let A and B be the two points, draw BB′ horizontal and AB′ vertical; let the body leave A with a velocity V, and arrive at B with a velocity v; then, if W be the weight of the body, the number of units of work accumulated in it while it moves from A to B, will equal (Ex. 644).

Fig. 144.



$$\frac{W}{2g}(v^2-V^2).$$

Now, the only pressures that have acted on the body are its weight, and the reaction of the curve; the work done by the former of these equals $W \times AB'$ (Art. 91 b) and the latter does no work, since its direction is always perpendicular to that in which its point of application is moving (Art. 92 b); therefore

$$\frac{W}{2g}(v^2-V^2) = W \times AB'$$

$$\therefore v^2-V^2 = 2g \times AB'$$

But this is the equation we should obtain if we supposed

a body to leave A with a velocity V, and to fall freely to B′. Therefore, &c. Q. E. D.

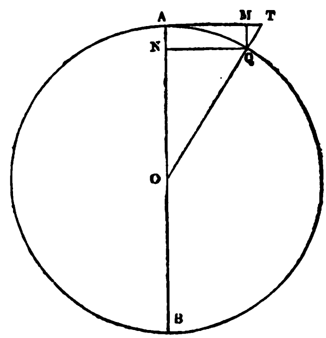
Cor.—The above proposition is true, whether AB is a plane curve, *e.g.* a circle, or a curve of double curvature, *e.g.* the thread of a screw. It will be an instructive exercise for the reader to make out the kind of effect which friction would have on the velocity in both these cases; the actual calculation requires the Integral Calculus.

Proposition 27.

If a heavy point whose weight is W be moving in a circle (whose radius is r) with a velocity V, the pressure (P) tending to the centre necessary to keep the body moving in the circle is given by the formula

$$P = \frac{W}{g} \cdot \frac{V^2}{r}.$$

Fig. 145.



Let A be the position of the point at the given instant, in the circle whose centre is O and diameter AB; at the end of a short time t, suppose the body to have come to Q; join OQ and produce it to meet in T, the tangent AT to the circle at A; draw QM parallel and QN at right angles to AB.

(1.) If while the point moves from A to Q, the pressure were to act continually parallel to AO, the velocity at A must be such as by itself would carry the point through the space AM in the time t, while the pressure must be such as would by itself drag the point through a space MQ in the same time (Prop. 24).

(2.) If while the point moves from A to Q, the pressure

were to act continually parallel to OQ, the velocity at A must be such as by itself would carry the point through the space AT in the time t, while the pressure must be such as would by itself drag the point through a space TQ in the same time (Prop. 24).

(3.) But since the pressure continually tends to O, the actual velocity V must be such as to carry the point in the time t through a space intermediate to AM and AT, while the actual pressure P must be such as in the same time to draw it through a space intermediate to QM and QT.

(4.) But ultimately $AT = AM$ and $QT = QM$

$$\therefore \; AM = Vt$$

$$\text{and } AN = \tfrac{1}{2} \cdot \frac{Pg}{W} t^2 * \text{ ultimately}$$

$$\therefore \; \tfrac{1}{2} \cdot \frac{Pg}{WV^2} = \text{the limit of } \frac{AN}{AM^2}$$

$$\text{now } \frac{AN}{AM^2} = \frac{1}{NB}$$

$$\therefore \text{ the limit of } \frac{AN}{AM^2} = \frac{1}{2r}$$

$$\therefore \; \tfrac{1}{2} \cdot \frac{Pg}{WV^2} = \frac{1}{2r}.$$

$$\text{or } P = \frac{W}{g} \cdot \frac{V^2}{r}.$$

Q. E. D.

Ex. 661.—A locomotive engine weighing 9 tons passes round a curve 600 yards in radius at the rate of 30 miles an hour; what pressure tending towards the centre of the curve must be exerted to make it move in this curve? *Ans.* 677·6 lbs.

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* If f is the acceleration of a body's velocity it describes from rest a space equal to $\frac{1}{2} ft.^2$ in t seconds, and if the body is acted on by a pressure P then $f = \frac{P}{W} g$. (Art. 110.)

Ex. 662.—If this pressure is supplied by making the inner rail on a lower level than the outer, what ought to be the difference of the level if the space between the rails is 4 ft. 9 in.? *Ans.* 1·9 in.

[The slope should be such that the resolved part of the weight along it shall equal the pressure determined in the last Example.]

Ex. 663.—On the floor of a railway carriage are chalked two lines xx', yy', one parallel and the other perpendicular to the direction of the rails; the lines intersect in the point O; at a height of 4 feet vertically over O is held a ball; the train moving at the rate of 30 miles per hour comes to a curve whose radius is 1000 ft. and centre in the prolongation of Ox'; if the ball is dropped, where will it strike the floor of the carriage?

Ans. In Ox at 5·8 in. from O.

Ex. 664.—A heavy point is tied to the end of a string whose length is l, it makes n revolutions per second; show that it will come into a position of steady motion when the string makes an angle θ with the vertical given by the equation

$$\cos\theta = \frac{g}{4\pi^2 n^2 l}.$$

[If T is the tension on the string, the vertical component of T must equal the weight of the body, and the horizontal component must be the pressure tending to the centre necessary to keep the body moving at the rate of n revolutions a second in a circle whose radius is $l \sin\theta$.]

Ex. 665.—A body weighing 12 lbs. is suspended by a cord 7 ft. long, and makes 80 revolutions per minute, determine the position of steady motion and the tension on the cord.

Ans. (1) 86° 15′ 55″ with the vertical. (2) 184¼ lbs.

Ex. 666.—A body weighing 20 lbs. is tied to the end of a string and suspended in a railway carriage the motion of which is perfectly steady; it comes to a curve 1000 ft. in radius, round which it runs at the rate of 15 miles an hour; find the inclination of the string to the vertical, and the horizontal pressure that would have to be applied to the body to keep the string vertical. *Ans.* (1) 52′. (2) 0·3025 lbs.

Ex. 667.—If the earth were a perfect sphere and at rest, so that the accelerating force of gravity at any point of its surface were g, show that the effect of its receiving its diurnal rotation will be to reduce the effective accelerating force of gravity to $g\left(1 - \frac{\cos^2 l}{289}\right)$ at a place whose latitude is l.

[See Ex. 594.—Latitude is measured from the equator.]

Ex. 668.—Given that the accelerating force of Jupiter's attraction on any point of its surface is 80 (in feet and seconds), that his radius is 11 times that of the earth, and that he makes one revolution in 10 hours, determine the ratio of the effective force of gravity at his equator to that at his pole.

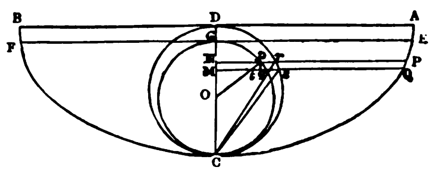
Ans. 0·913 nearly.

Proposition 28.

If a is the radius of the generating circle of a cycloid* which is placed in a vertical plane with its vertex downward and base horizontal, the time in which a body falls along it freely from any point to the lowest equals $\pi\sqrt{\frac{a}{g}}$.

Let ACB be the cycloid, CD the vertical diameter of the generating circle, E the point from which the body is allowed to fall; draw the horizontal line EF, cutting CD in G, and on CG as a diameter describe a circle the centre of which is O; at the end of any time t let the body have reached P; draw PN, QM at right angles to CD, cutting the circles on p, r and q, s respectively; join Cr, Cs, Op; draw pt at right angles to CM.

Fig. 146.



Now, the velocity the body has acquired at P equals $\sqrt{2g.\,\mathrm{GN}}$ (Prop. 26); therefore if δt is the time in which it passes over the arc PQ, we have

$$\delta t = \frac{\mathrm{PQ}}{\sqrt{2g\,.\,\mathrm{GN}}} \text{ ultimately.}$$

Since ultimately we may conceive the body to move uniformly during the time δt.

Also we have PQ (see Appendix I.) $=2\,(\mathrm{C}r - \mathrm{C}s)$

$$= \frac{2\,(\mathrm{C}r^2 - \mathrm{C}s^2)}{\mathrm{C}r + \mathrm{C}s} = \frac{\mathrm{C}r^2 - \mathrm{C}s^2}{\mathrm{C}r} \text{ ult.}$$

* See Appendix I.

But $Cr^2 = DC.CN$ and $Cs^2 = DC.CM$ and $DC = 2a$

$$\therefore PQ = \frac{2a\,(CN - CM)}{\sqrt{2a\,.\,CN}} = \frac{NM\sqrt{2a}}{\sqrt{CN}} \text{ ult.}$$

$$\therefore \delta t = \frac{\sqrt{2a}\ .\ MN}{\sqrt{2g\,GN.NC}} = \sqrt{\frac{a}{g}}\cdot\frac{MN}{Np}, \text{ since } Np^2 = GN.NC.$$

Now, ultimately pq coincides with the tangent to the circle at p, therefore pqt is ultimately a right-angled triangle whose sides pq and pt are severally perpendicular to Op and pN; therefore by similar triangles

$$pq : pt :: Op : Np$$

$$\therefore \frac{pq}{Op} = \frac{pt}{Np} = \frac{MN}{Np}$$

$$\therefore \delta t = \sqrt{\frac{a}{g}}\cdot\frac{pq}{Op} \text{ ultimately,}$$

and the same being true of every interval of time while the body falls from E to C, the whole time, which is the limit of the sum of the intervals when their number becomes great, will equal the sum of the ultimate values of δt, *i.e.* it will equal $\sqrt{\frac{a}{g}}\cdot\frac{GpC}{Op}$ or $\pi\sqrt{\frac{a}{g}}$

Q. E. D.

Cor.—If a heavy point were suspended by a perfectly flexible string from a point in the prolongation of CD, and were constrained to move in the cycloid, it would move at each point in the same manner as if it were falling down the curve; also it is evident that the time of ascending the arc CF must exactly equal that of descending the arc EC, and therefore the time of one cycloidal oscillation must equal $2\pi\sqrt{\frac{a}{g}}$; but, when the point is thus con-

strained to move in a cycloid, the length l of the thread equals $4a$ (see Appendix I.); therefore the time of one cycloidal oscillation is $\pi\sqrt{\frac{l}{g}}$.

Ex. 669.—Show that the time of a *small* oscillation in a circle whose radius is l will equal $\pi\sqrt{\frac{l}{g}}$.

[The oscillation is *small* if we may consider the semi arc of vibration equal to its chord. In fig. to Prop. 28 suppose EC to be a circular arc, omit the generating circle; we have $PQ = CP - CQ = \frac{CP^2 - CQ^2}{CP + CQ}$ and this equals $\frac{l \,.\, MN}{\sqrt{2l \,.\, CN}}$, if we consider the arcs equal to chords; the remainder of the proof is the same as in Prop. 28.]

Ex. 670.—What is the length of a simple pendulum which at Greenwich oscillates in $1\frac{1}{2}$ seconds? How much shorter is the simple pendulum which at Rawak oscillates in the same time? *Ans.* (1) 7·3387 ft. (2) 0·2824 in.

Ex. 671.—A pendulum whose length is L makes m oscillations in one day; its length changes, and it is now observed to make $m+n$ oscillations in one day, show that its length has been diminished by a part equal to $\frac{2n}{m}$ L (nearly).

[Since a mean solar day contains 86,400 seconds, we have

$$\frac{86400}{m} = \pi\sqrt{\frac{L}{g}} \text{ and } \frac{86400}{m+n} = \pi\sqrt{\frac{L-\delta L}{g}}$$

$$\therefore \frac{m+n}{m} = \sqrt{\frac{L}{L-\delta L}} \text{ whence } \delta L = \frac{2n}{m} L.]$$

Ex. 672.—A pendulum in a certain place makes in one day m oscillations, on transporting it to another place it is found to have the same length but to lose n oscillations a day; show that the force of gravity has been diminished by its $\frac{2n}{m}$th part.

Ex. 673.—Given the lengths of the seconds pendulums at Greenwich and Paris respectively (see Table XV.), find how many oscillations a day the Greenwich pendulum would make at Paris. *Ans.* 86387.

Ex. 674.—Given that a pendulum oscillating seconds at the mouth of a coal pit gains 2·24 per diem when removed to the bottom of the shaft; determine the decrease of the accelerating force of gravity. *Ans.* 0·0016.

Ex. 675.—If a heavy point vibrates in a cycloidal arc and begins to move at the end of an arc whose length from the lowest point is s_1, and if T is

the time in which it moves over the whole arc of vibration show that if it reaches a distance s from the lowest point at the end of a time t, then

$$s = s_1 \cos \frac{\pi t}{T}.$$

[In Prop. 28 the time of falling from E to P equals $\sqrt{\frac{a}{g}} \cdot \frac{Gp}{Op}$]

Ex. 676.—If a body vibrates in the whole arc of a cycloid show how to divide each half into n parts which shall be described in equal times, and show that their lengths beginning from the lowest are respectively, $s \sin \frac{\pi}{4n} \cdot \cos \frac{\pi}{4n}$, $s \sin \frac{\pi}{4n} \cdot \cos \frac{3\pi}{4n}$, $s \sin \frac{\pi}{4n} \cdot \cos \frac{5\pi}{4n}, \ldots$ $s \sin \frac{\pi}{4n} \cdot \cos \frac{(2p-1)\pi}{4n}, \ldots$ where s is the whole length of the cycloid.

Ex. 677.—If a point oscillates in a cycloid or in a *small* circular arc (length of thread being equal to l), show that the acceleration along the curve at a distance s measured from the lowest point along the curve is $\frac{g}{l} \cdot s$; and hence show that if a body whose weight is W vibrates under the action of a pressure Hs, where s is the distance of W from the middle point of the vibration, the time of each vibration will be given by the formula $\pi \sqrt{\frac{W}{Hg}}$.

125. *Longitudinal vibrations of a rod.*—If there is a rod whose length is L, area of section A, and modulus of elasticity E, and if to the end of it is attached a weight Q (which we will suppose to be so large that the weight of the rod can be neglected), then if the rod is allowed to lengthen slowly, Q will descend through a small space l equal to $\frac{LQ}{KE}$ and will continue at rest (see Art. 6, and Ex. 149); if, however, it is allowed to descend at once, a certain number of units of work will be accumulated in it when Q has descended through the space l, so that it will continue to descend till the resistance to further elongation shall have destroyed them, and then a contraction will ensue, and thus Q will vibrate in a vertical line about the point (A), at which in the former case it would have come to rest.

Ex. 678.—Show that when the weight Q is at a distance s from A it is moving under a pressure that varies as s, and that the time in which it proceeds from the highest to the lowest point is $\pi\sqrt{\frac{QL}{gKE}}$.

Ex. 679.—In the last Example suppose Q to be at a distance s below A, determine the number of units of work accumulated in it at that instant, and show that its velocity (V) is given by the equation

$$V^2=\frac{g}{l}\,(l^2-s^2).$$

[See Ex. 149. Compare the value of V^2 with Ex. 589.]

Ex. 680.—If a cylinder whose height is h and specific gravity s floats with its axis vertical in a fluid whose specific gravity is s_1, show that if it is depressed through *any* distance the time in which it will rise from its point of greatest depression to its greatest height is constant, and will be given by the formula

$$t=\pi\sqrt{\frac{h}{g}\cdot\frac{s}{s_1}}.$$

CHAP. V.

THE MOMENT OF INERTIA.

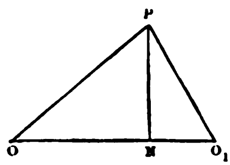
Def. — If we conceive a body to consist of a large number of heavy points, and multiply the mass of each by the square of its perpendicular distance from a given line or axis, the sum of all these products is the moment of inertia of the body with respect to that axis.

126. *Properties of the moment of inertia.*—It will appear hereafter that the moment of inertia is a quantity that enters nearly every question in which the rotatory motion of a body is concerned; the present chapter will be devoted to proving some of its properties, and ascertaining its magnitude in certain particular cases. The first property we shall notice is one that follows immediately from the definition. Since the mass of a particle and the square of its perpendicular distance from a given axis are essentially positive, their product must be so too; consequently if we conceive any group of heavy points to consist of two or more subordinate groups, the sum of the moments of inertia of these separate groups with respect to a given axis will equal that of the whole group with respect to the same axis: hence if a body can be divided into a certain number of parts, and their moments of inertia are known with respect to a certain axis, that of the whole body, with respect to that axis, is found by adding them together.

Proposition 29.

If I is the moment of inertia of any body whose weight is W, about an axis passing through its centre of gravity, and I_1 the moment of inertia of the same body about a parallel axis situated at a perpendicular distance h from the former, then

Fig. 147.



$$I_1 = I + \frac{W}{g} h^2.$$

Suppose the axes to be perpendicular to the plane of the paper, and let the axis which passes through the centre of gravity meet that plane in O, and the other meet it in O_1; let P be one of the points of which the body is conceived to be made up, and let its mass be $\frac{w_1}{g}$; join PO, PO_1, and OO_1, and draw PN perpendicular to OO_1; then Eucl. II. 13)

$$O_1P^2 = OP^2 + OO_1^2 - 2OO_1 . ON.$$

Let $OP = r_1$, $O_1P = r'_1$, $ON = x_1$, and $OO_1 = h$, then

$$\frac{w_1}{g} . r_1'^2 = \frac{w_1}{g} r_1^2 + \frac{w_1}{g} . h^2 - 2 \frac{w_1}{g} h x_1 \qquad (1)$$

and the same algebraical formula will be true whatever be the position of P; hence if $w_2\ r'_2\ r_2\ x_2$, $w_3\ r'_3\ r_3\ x_3$, &c.... are the magnitudes corresponding to other points, we shall have

$$\frac{w_2}{g} r'^2_2 = \frac{w_2}{g} r_2^2 + \frac{w_2}{g} h^2 - 2 \frac{w_2}{g} hx_2 \qquad (2)$$

$$\frac{w_3}{g} r'^2 = \frac{w_3}{g} r_3^2 + \frac{w_3}{g} h^2 - 2 \frac{w_3}{g} hx_3 \qquad (3)$$

and so on for every point.

Now by the definition

$$w_1 r'^2_1 + w_2 r'^2_2 + w_3 r'^2_3 + \ldots = g\,\mathrm{I}_1$$
$$w_1 r_1^2 + w_2 r_2^2 + w_3 r_3^2 + \ldots = g\,\mathrm{I}$$

also,
$$w_1 + w_2 + w_3 + \qquad \ldots = \mathrm{W}$$

and by the properties of the centre of gravity (Prop. 15)

$$w_1 x_1 + w_2 x_2 + w_3 x_3 \ldots = 0$$

Therefore by adding the equations (1), (2), (3), &c., we obtain

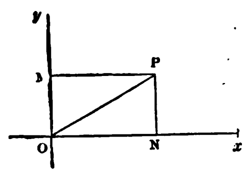
$$\mathrm{I}_1 = \mathrm{I} + \frac{\mathrm{W}}{g}h^2. \qquad \text{Q. E. D.}$$

Proposition 30.

If I_1 and I_2 are respectively the moments of inertia of any system of points lying in a plane about two rectangular axes in that plane, and if I is its moment of inertia about an axis perpendicular to the two others, and passing through their point of intersection, then

$$\mathrm{I} = \mathrm{I}_1 + \mathrm{I}_2$$

Fig. 148.



For let Ox, Oy be the two axes, the third being perpendicular to the plane of the paper, and passing through O; let P be one of the points whose mass is $\frac{w}{g}$; draw PM and PN perpendicular to Oy and Ox, join OP, and let PM$=x$, PN$=y$, OP$=r$, then

$$r^2 = x^2 + y^2$$
$$\therefore\ \frac{w}{g}r^2 = \frac{w}{g}x^2 + \frac{w}{g}y^2 \qquad (1)$$

Similarly, if other points are taken, and the corresponding

magnitudes are $w_1\, r_1\, x_1\, y_1$, $w_2\, r_2\, x_2\, y_2$, ..., we shall have

$$\frac{w_1}{g} r_1^2 = \frac{w_1}{g} x_1^2 + \frac{w_1}{g} y_1^2 \qquad (2)$$

$$\frac{w_2}{g} r_2^2 = \frac{w_2}{g} x_2^2 + \frac{w_2}{g} y_2^2 \qquad (3)$$

and so on, whatever be the number of points. Now

$$\frac{w}{g} r^2 + \frac{w_1}{g} r_1^2 + \frac{w_2}{g} r_2^2 + \ldots. = \mathrm{I}$$

$$\frac{w}{g} x^2 + \frac{w_1}{g} x_1^2 + \frac{w_2}{g} x_2^2 + \ldots. = \mathrm{I}_1$$

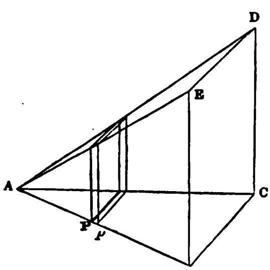
$$\frac{w}{g} y^2 + \frac{w_1}{g} y_1^2 + \frac{w_2}{g} y_2^2 + \ldots. = \mathrm{I}_2$$

Therefore, adding together (1), (2), (3), &c., we obtain

$$\mathrm{I} = \mathrm{I}_1 + \mathrm{I}_2. \qquad \text{Q. E. D.}$$

Ex. 681.—If k is the area of the section of a thin rod, w the weight of a cubic foot of the material, and l its length, show that its moment of inertia about an axis passing through one end and perpendicular to it equals $\frac{1}{3}\frac{w}{g}.kl^3$.

Fig. 149.



[If AB is the line, and a pyramid is constructed whose base BD is a square the side of which equals AB and its plane perpendicular to AB (compare Art. 64); then if we consider a lamina contained by planes drawn parallel to the base through the extremities of any small portion Pp of AB, its volume will ultimately equal Pp × AP²; now the moment of inertia of Pp equals mass of Pp × AP², *i. e.* it equals $\frac{wk}{g}$ × vol. of lamina; and hence the moment of inertia of the rod equals $\frac{wk}{g}$ × volume of the pyramid.]

Ex. 682.—The moment of inertia of the rod in the last Example about an axis perpendicular to its length and passing through its middle point equals $\frac{1}{12} \cdot \frac{w}{g} k l^3$.

[See Prop. 29.]

Ex. 683.—There is a rectangular lamina whose thickness is k and sides a and b, show that with reference to an axis parallel to a and passing through the middle point of b the moment of inertia equals $\frac{1}{12} \cdot \frac{w}{g} k\,ab^3$.

[See Art. 126.]

Ex. 684.—If in the last Example the axis is perpendicular to the plane and passes through the centre of gravity, show that the moment of inertia of the lamina equals $\frac{1}{12} \cdot \frac{w}{g} k\,ab\,(a + b^2)$.

[See Prop. 30.]

Ex. 685.—There is a rectangular parallelopiped whose edges are a, b, c, an axis is drawn through the centre of gravity and parallel to the edge c, show that the moment of inertia about that axis equals $\frac{1}{12} \cdot \frac{w}{g} abc\,(a^2 + b^2)$.

[See Art. 126.]

Ex. 686.—There is a right prism whose base is a right-angled triangle, the sides containing the right angle of which are a and b, the height of the prism is c. Show that if an axis be drawn through the centres of gravity of the ends the moment of inertia about that axis equals $\frac{1}{36} \cdot \frac{w}{g} abc \times (a^2 + b^2)$.

[By Art. 126 and Ex. 684 the moment of inertia about a parallel axis joining the middle points of the hypotheneuses of the ends is $\frac{1}{24} \cdot \frac{w}{g} abc \times (a^2 + b^2)$; the result is then obtained by Prop. 29.]

Ex. 687.—There is a right prism whose height is c and base an isosceles triangle the base of which is a and height b; if an axis be drawn passing through the centres of gravity of the ends its moment of inertia about that axis equals $\frac{1}{12} \cdot \frac{w}{g} abc \left(\frac{a^2}{4} + \frac{b^2}{3}\right)$.

[This prism can be divided into two resembling that in the last Ex.]

Ex. 688.—There is a right prism whose weight is W and base a regular polygon, the radius of whose inscribed circle is r, and length of one side a, show that its moment of inertia about its geometrical axis is $\frac{1}{2} \cdot \frac{W}{g} \times \left(\frac{a^2}{12} + r^2\right)$.

[This prism can be divided into prisms like that in the last Example.]

Ex. 689.—If there is a cylinder whose height is h and radius of base r, show that its moment of inertia about its geometrical axis equals $\frac{\pi}{2} \frac{w}{g} h r^4$

[If the cylinder reduces to a circular lamina whose thickness is h, the same formula is of course true.]

Ex. 690.—There is a thin circular lamina whose radius is r and thickness k, show that the moment of inertia about a diameter equals $\frac{\pi}{4}\,\frac{w}{g}\,k r^4$.

[See Prop. 30.]

Ex. 691.—There is a drum the length of which is a, the mean radius of the end r, and the thickness t, show that its moment of inertia about its axis very nearly equals $\frac{w}{g}\,2\pi\,at\,r^3$; and that if t equals $\frac{r}{n}$, the error in the above determination of the moment of inertia is the $\frac{1}{4n^2}$th part of that quantity.

Ex. 692.—There is a cylinder the length of which is h and the radius of whose base is r, show that its moment of inertia about a diameter of one end equals $\frac{w}{g}\,\pi h r^2 \left\{\frac{h^2}{3}+\frac{r^2}{4}\right\}$.

[If we consider a lamina contained between two planes parallel to the end and at distances x and $x+\delta x$, it appears from Ex. 690 and Prop. 29 that the moment of inertia of the lamina equals $\frac{1}{4}\,.\,\frac{w}{g}\,\pi r^4\delta x+\frac{w}{g}\pi r^2 x^2\delta x$; whence the required moment of inertia equals the mass of a line each foot of which weighs $\frac{1}{4}\,.\,w\pi r^4$, together with the moment of inertia about one end of a line every foot of which weighs $w\pi r^2$.]

Ex. 693.—Determine the moment of inertia of a cylinder about a generating line.

*Ex. 694.—There is a cone the height of which is h and radius of base r, show (1) that its moment of inertia about its axis equals $\frac{1}{10}\,\frac{w}{g}\,\pi h r^4$; (2) that its moment of inertia about an axis drawn through the vertex and perpendicular to the axis of the cone equals $\frac{1}{5}\,\frac{w}{g}\,\pi h r^2\left\{h^2+\frac{r^2}{4}\right\}$.

*Ex. 695.—Show that the moment of inertia of a sphere about any diameter equals $\frac{8}{15}\,\frac{w}{g}\pi r^5$.

[The results in the last two Examples cannot be easily obtained without the aid of the integral calculus.]

Ex. 696.—In the mass of iron described in Ex. 12 let an axis be drawn passing through the end of the longer rectangular piece and bisecting those sides of the end which are 6 inches long, determine the moment of inertia of the mass with respect to that axis.* *Ans.* 2309·262.

Ex. 697.—There is a cast iron cone 16in. high, radius of base 8in., deter-

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* In this, as in all examples of moments of inertia, weight is reckoned in lbs. and space in feet.

mine its moment of inertia, (1) about an axis through its centre of gravity and parallel to its base, (2) about a parallel axis distant 4 feet from the former. *Ans.* (1) 1·1647. (2) 140·9267.

Ex. 698.—Determine the moment of inertia about a vertical edge of the oak door described in Ex. 17. *Ans.* 14·4047.

Ex. 699.—There is a cube of oak whose edge is 8 inches long, through the middle of it at right angles to one of its faces passes a cylinder of oak 4 feet long and 3 inches in diameter; the centres of gravity of the two figures coincide; determine the moment of inertia of the whole about an axis passing through the common centre of gravity and perpendicular to the axis of the cylinder and also to a face of the cube. *Ans.* 0·5166.

Ex. 700.—Determine the moment of inertia of the hollow leaden cylinder described in Ex. 15 about a diameter of its mean section.

Ans. 0·020194.

Ex. 701.—If a cylinder like that in the last Example is fitted to each arm of the figure described in Ex. 699, determine the moment of inertia of the whole about the specified axis, (1) when the ends of the leaden cylinders coincide with those of the arms, (2) when the other ends of the cylinders are in contact with the cube. *Ans.* (1) 6·2966. (2) 0·9.

Ex. 702.—Determine the moment of inertia about the axis of a grindstone 4 feet in diameter and 8 inches thick. *Ans.* 70·13.

Ex. 703.—There is a cast iron flywheel consisting of a rim, four spokes at right angles to each other, and an axle; the external and internal radii of the rim are 4 and $3\frac{1}{2}$ ft. respectively, and its thickness 8 in.; the sections of the spokes are each 4 square inches, the axle 12 in. in diameter and 18 in. long: determine the moment of inertia of the whole about the geometrical axis of the axle; and also determine the error if the spokes and axle were neglected and the moment of inertia of the rim calculated by Ex. 691.

Ans. (1) 1586. (2) 31·28.

Ex. 704.—If the moment of inertia of any body with reference to any axis be represented by I, and if the body be uniformly expanded by heat, so that the linear dimensions before expansion are to those after in the ratio of $1 : 1 + a$, show that the moment of inertia with reference to the given axis becomes $(1 + a)^5$ I.

127. *The radius of gyration.*—It is evident from the definition of the moment of inertia of a body with respect to a given axis, that there will be, with respect to that axis, a line of a certain determinate length k, such that

$$I = \frac{Wk^2}{g}$$

where I is the moment of i ertia, and W the weight of

the body; the line k is called the radius of gyration with respect to that axis, and may be defined to be that distance from the axis at which the whole mass of the body may be supposed to be collected without producing any change in the moment of inertia. Thus, it is evident that in Ex. 681, 684, and 689, the values of the radius of gyration are respectively $\frac{l}{\sqrt{3}}$, $\sqrt{\frac{a^2+b^2}{12}}$ and $\frac{r}{\sqrt{2}}$. Moreover, if k be the radius of gyration of a body with reference to an axis passing through the centre of gravity, and k_1 its radius of gyration with reference to an axis parallel to the former, and at a distance from it equal to h, then it is evident from Prop. 29 that

$$k_1^2 = k^2 + h^2.$$

It is to be observed that the moment of inertia is essentially a mechanical magnitude, while the radius of gyration is simply a line; now suppose k to be the radius of gyration of any lamina, the area of the face of which is A, it is not unusual to speak of that *area* as having a moment of inertia; when this is done it means that

$$I = Ak^2.$$

In this sense the term moment of inertia is used in Art. 88. Strictly speaking, an *area* no more has a moment of inertia than it has weight.

CHAP. VI.

D'ALEMBERT'S PRINCIPLE.

128. *Account of problem to be solved.*—The manner in which the dimensions of a body influence its motion may be illustrated as follows:—If we suppose a bar to be suspended by one end and to oscillate, the velocities with which the different points are, at any instant, moving, stand to one another in a fixed relation; thus the end moves twice as fast as the middle point; moreover, with one exception, each point has a different velocity from what it would have if it were detached from the rest, and swang freely at the same distance from the centre of suspension; this difference must depend upon the cohesive forces which bind the parts of the bar together. The consideration of this simple case points out the two chief additional conceptions required for the investigation of the motion of a body whose form has to be taken into account.

(1.) A means must be obtained for comparing the velocities of different points of a rigid body revolving round an axis, which is done by introducing the conception of *Angular Velocity.*

(2.) A principle is required by means of which we can avoid the consideration of the cohesive forces which hold together the parts of the body: this is generally called D'Alembert's Principle.

129. *Angular velocity.*—If a rigid body revolves round an axis, it is plain that the perpendiculars let fall from each

point of the body on the axis will, in a given time, describe equal angles; hence arises the following

Def.—If a body revolves uniformly round an axis, the angle (estimated in *circular measure*) described in one second by the perpendicular let fall from any point on the axis of rotation is called the *angular velocity* of the body.

If the velocity is variable, it is measured at any instant by the angle that would be so described if, from that instant, the velocity continued uniform for one second.

In the following pages ω and Ω are used to denote angular velocity.

Ex. 705.—A body makes 30 uniform revolutions in one minute, what is its angular velocity? *Ans.* π.

Ex. 706.—A body moves at the rate of 12 ft. per second in a circle whose radius is 15 ft., what is its angular velocity? *Ans.* $\frac{4}{5}$.

Ex. 707.—Determine the angular velocity of the earth round its axis. *Ans.* $\frac{\pi}{42082}$.

[See Example 528.]

Ex. 708.—If a body has an angular velocity 2·5, how many revolutions will it make per hour? *Ans.* 1432·4.

Ex. 709.—If a body has a uniform angular velocity ω, show that the centrifrigal force of a point in it, situated at a distance r from the axis, is $r\omega^2$.

130. *Impressed forces.*—All forces acting on a body which do not arise out of the mutual cohesion of its parts, are called the *impressed forces* that act on the body.

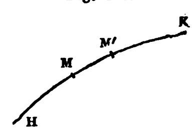
Thus, when a cricket ball is thrown in vacuo, the impressed force is gravity; if it were pierced by a spindle and caused to revolve round it, the impressed force would be gravity and the reaction of the spindle; and so on in other cases.

131. *Effective forces.*—It must be remembered that when a solid body is in motion, each point in it moves along a determinate line, straight or curved according to circumstances. As this fact should be distinctly conceived by the student, it may be mentioned by way of

illustration that, when a cart moves along a perfectly even road, each point on the circumference of one of its wheels describes a cycloid, the centre of the wheel describes a straight line, while any point in one of the spokes describes a curve called a trochoid. A similar, though much more complicated, kind of motion belongs to the different points of a cricket ball, when in the act of being thrown it receives a rotatory motion. The only fact, however, that we are concerned with here is that, whatever be the motion of the body, each point in it will describe a *determinate* path.

Let w be the weight of a point of a moving body, and suppose that point to describe the path HK; at M let it be moving with a velocity v, and at M′ with a velocity v', having described the small space MM′ in the short time t. Let it now be inquired what pressures acting on an isolated point would make it move as the point actually does when forming part of the moving body. The points MM′ may be considered to be on the circumference of the circle of curvature at the point M, whose radius ρ can be determined from the nature of the curve HK; hence at M the isolated point must be acted on by a normal pressure equal to $\frac{w}{g} \cdot \frac{v^2}{\rho}$, and the change of velocity must be produced by a tangential pressure $\frac{w}{g} \cdot \frac{v'-v}{t}$, the time t being supposed indefinitely small. If, then, at M we suppose the point to become isolated, retaining its velocity and direction, it will continue to move as it actually does during the next short time, if acted on by the resultant of the pressures $\frac{w}{g} \cdot \frac{v^2}{r}$, and $\frac{w}{g} \cdot \frac{v'-v}{t}$; this resultant is called the effective force,

Fig. 150.



or the effective pressure on the particle at M. Hence we may define it in general terms as follows:—

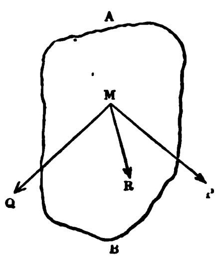
Def.—If the velocity and direction of a point, forming part of a rigid body, undergoes a certain change in an indefinitely short time beginning at a given instant; then if we suppose the point to be at that instant disconnected from the body, and to be acted on by a pressure which produces in that indefinitely short time the same change in its velocity and direction, the pressure is called the effective pressure, or the effective force on the point at that instant.

132. *Effective pressures in the case of rotatory motion.*—Suppose a point whose mass is $\frac{w}{g}$ to be situated at a distance r from the axis of rotation of a body, of which the point forms a part; let ω be the angular velocity of the body at a given instant, and at the end of a short time t let the angular velocity become ω', then at the given instant the effective pressure will consist of two components $\frac{w}{g}\,r\omega^2$ along r, and $\frac{w}{g}\cdot\frac{r(\omega'-\omega)}{t}$ in the direction of the tangent; if the angular velocity is uniform, the second component is zero, and the effective pressure is $\frac{w}{g}\,r\omega^2$ acting along r.

133. *D'Alembert's principle.*—Let it now be inquired what are the pressures that act on the point M of the moving body AB; it will be remarked that they can only be of two kinds, (1) the impressed pressure P transmitted to it, (2) the resultant Q of the cohesive pressures which bind it to the rest of the body: these two pressures must have at any given instant a determinate resultant R, and this must be the effective pressure on M at that instant, since if M were isolated for a short time, and were acted on by R, it would experience the same change in velocity

and direction that it actually experiences; now if a pressure equal and opposite to R were to act on M at the instant under consideration, it would be in equilibrium with P and Q; and the same is true of every other point of the body; consequently if we suppose pressures equal and opposite to the effective pressure of each point to be applied to points of the body respectively, they will be in equilibrium with the impressed and cohesive pressures, and we shall have three systems of pressures constituting a system in equilibrium, viz. (1) a system of impressed pressures, (2) a system of cohesive pressures, (3) a system of effective pressures applied to the points in the opposite direction to that in which they must act to produce the actual motion of the points. Now D'Alembert's principle asserts that the cohesive pressures are separately in equilibrium, and infers the conclusion that *if pressures equal and opposite to the effective pressures at any instant were at that instant applied to each point of the body, they would be in equilibrium with the impressed pressures.*

Fig. 151.



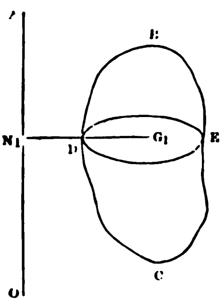
Proposition 31.

If a body, whose mass equals $\frac{W}{g}$, is symmetrical with reference to a plane passing through a certain axis and its centre of gravity, the distance of which from the axis is denoted by $\bar{x}$; then if the body revolves round the axis with a uniform angular velocity ω, the resultant of the effective pressures equals $\frac{W}{g}\bar{x}\omega^2$.

Let AO be the axis, and BC the revolving body, the plane of the paper being the plane of symmetry, we

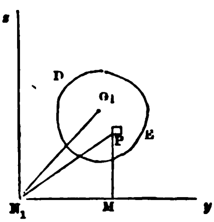
may suppose it divided into a number of laminæ, such as DE, by planes perpendicular to AO; then if we find the effective pressure of each lamina, their resultant will be the required pressure.

Fig. 152.



(1.) Let G_1 be the centre of gravity of DE, and $G_1 N_1$ its perpendicular distance (x_1) from the axis AO; in its plane draw the axes $N_1 y$, $N_1 z$, and take P, any small portion of it, and suppose the mass of P to be $\frac{w}{g}$, and its co-ordinates to be y and z; also let the angle $PN_1 y$ be denoted by θ. Now (Art. 132) since P describes a circle round N_1 with a uniform velocity, its effective pressure is $\frac{w}{g}\omega^2 . PN_1$ which can be resolved into two components, viz. $\frac{w}{g}\omega^2 y$ parallel to $N_1 y$, and $\frac{w}{g}\omega^2 z$ parallel to $N_1 z$. In the same manner, if $\frac{w_1}{g}$, y_1, z_1, $\frac{w_2}{g}$, y_2, z_2, ... are the corresponding values for other elements of the lamina, we shall have pressures $\frac{w_1}{g}\omega^2 y_1$, $\frac{w_2}{g}\omega^2 y_2$ . . . . parallel to $N_1 y$, and $\frac{w_1}{g}\omega^2 z_1$, $\frac{w_2}{g}\omega^2 z_2$ . . . . parallel to $N_1 z$. Hence (Prop. 15) the effective pressures on the lamina are equivalent to the two

Fig. 153.



$$\frac{\omega^2}{g}\left\{wy + w_1 y_1 + w_2 y_2 + \ldots\right\} = \frac{\omega^2}{g} W_1 \bar{y} \text{ parallel to } N_1 y$$

$$\text{and } \frac{\omega^2}{g}\left\{wz + w_1 z_1 + w_2 z_2 + \ldots\right\} = \frac{\omega^2}{g} W_1 \bar{z} \text{ parallel to } N_1 z,$$

where $\bar{y}\ \bar{z}$ are the co-ordinates of G_1 and W_1 is the weight of the lamina DE; if we compound these two pressures, we shall obtain $\frac{\omega^2}{g} . W_1 x_1$ as their resultant acting along $G_1 N_1$.

(2.) Let the masses of the several laminæ into which BC is divided be respectively $\frac{W_1}{g} . \frac{W_2}{g} . \frac{W_3}{g} \ldots$ and the respective distances of their centres of gravity from AO be $x_1, x_2, x_3 \ldots\ldots$ then their effective pressures are severally $\frac{\omega^2}{g} W_1 x_1, \frac{\omega^2}{g} W_2 x_2, \frac{\omega^2}{g} W_3 x_3, \ldots$ Now it follows from the symmetry of the figure, that all these centres of gravity are in the same plane, viz. the plane of the paper; the effective pressures are therefore parallel, and their resultant will equal their sum, viz.

$$\frac{\omega^2}{g}(W_1 x_1 + W_2 x_2 + W_3\ x_3 + \ldots)$$

which equals $\frac{\omega^2}{g} W\bar{x}$ or $\frac{W}{g} \bar{x} \omega^2$. Q. E. D.

Cor.—The point at which the direction of the resultant of the effective pressures cuts the axis is determined thus: take any point O on the axis, and let ON_1 be denoted by z_1, and let $z_2, z_3 \ldots$ correspond to the other laminæ, then if $\bar{x}$ is the distance of the required point from O, we have (Prop. 14.)

$$\frac{W}{g} \omega^2 \bar{x}\ \bar{z} = \frac{\omega^2}{g} \left\{ W_1 x_1 z_1 + W_2 x_2 z_2 + W_3 x_3 z_3 + \ldots \right\}$$

Now in general the right hand side of this equation cannot* be obtained except by means of the integral

* The right hand side of the equation is commonly written $\omega^2 \Sigma mzx$; and it may be added that Σmzx is one of the three quantities Σmxy, Σmyz, Σmzx that occur in systematic treatises on the dynamics of a solid body.—See Poisson, Mécanique, vol. ii. c. 2.

calculus: one important exception, however, may be mentioned, viz. when the body is symmetrical with reference to a plane perpendicular to the axis of rotation; in this case it is evident that if we take O at the point where this plane cuts the axis, the right hand side of the above equation will equal zero, *i. e.* the pressure P must act along the intersection of two planes of symmetry, so that the direction of the resultant of the effective pressures must pass through the centre of gravity of the body. Examples of this case are supplied by a sphere revolving round any axis, a cylinder revolving round an axis either parallel or perpendicular to its geometrical axis, and a cone about an axis perpendicular to its geometrical axis.

Ex. 710.—A thin rod whose length is l is fastened by one end to a spindle, to which it is inclined at an angle α and round which it revolves, show that the direction of the resultant of the effective pressures cuts the spindle at a distance from the end equal to $\frac{2}{3} l \cos \alpha$.

Ex. 711.—A cone of cast iron 1 ft. high, the radius of whose base is 6 in., revolves 30 times a minute round an axis parallel to its geometrical axis, and passing through a point in the circumference of the base; find the centrifugal force, *i. e.* the resultant of the effective pressures. *Ans.* 18·2 lbs.

Ex. 712.—A cylinder of cast iron 3 ft. high, whose base is 6 in. in diameter, revolves 100 times a minute with its axis vertical round a parallel axis at a distance of $1\frac{1}{2}$ ft.; find the centrifugal force. *Ans.* 1364 lbs.

Ex. 713.—A wrought iron rod 10 ft. long and section 1 in. in radius is made to revolve 60 times in a minute round an axis perpendicular to its length and passing through one extremity; find the centrifugal force.

Ans. 606 lbs.

Ex. 714.—Two balls of cast iron, one 10 in. and the other 6 in. in diameter, are fixed to the ends of a rod with their centres 3 ft. apart; they are made to revolve 100 times a minute about a vertical spindle, whose distance from the centre of the heavier ball is 1 ft.; find the pressure due to centrifugal force on the spindle. *Ans.* 266 lbs.

Ex. 715.—Two rods in all respects equal are made to revolve about a vertical spindle; they are always in the same vertical plane but on different sides of the spindle, and are quite free to move round the top of the spindle in that plane; if the spindle makes n revolutions per second determine the position of steady motion. *Ans.* $\cos \alpha = \frac{3g}{8\pi^2 n^2 l}$

Ex. 716.—A shaft of cast iron whose section is 8 in. by 4 in. and whose

length is 4 ft., revolves in a horizontal plane round a vertical axi of wrought iron 6 in. in diameter whose centre is 4 in. from the end of the shaft; if it makes 200 revolutions per minute, determine the number of units of work expended on the friction of the axle caused by the centrifugal force, the axle being well greased ($\mu = 0{\cdot}075$). *Ans.* 215590 units per min.

134. *Pressure on a fixed axis of rotation.*—The student must be on his guard against supposing that $\frac{W}{g}\bar{x}\omega^2$ is the whole of the pressure on the fixed axis; though it is frequently the most important part of it. The complete investigation of that pressure lies beyond the scope of the present work; to prevent a misapprehension, however, it may be well to add one or two of the results of the investigation.

(1.) The body being supposed symmetrical, as in Prop. 31, and it being further supposed that no external force, such as gravity, acts upon the body, the only impressed force will be the reaction of the axis, which (Art. 133) will therefore equal $\frac{W}{g}\bar{x}\omega^2$.

(2.) If in the last case the axis were vertical, and the body acted on by gravity, the horizontal pressure is still $\frac{W}{g}\bar{x}\omega^2$; but there is also a vertical pressure acting along the axis equal to the weight of the body.

(3.) If in the last case the body were not symmetrical with reference to a plane passing through the axis and the centre of gravity, there will in general be the following pressures: (*a*) a pressure equal to the weight of the body acting along the axis; (*b*) in the plane passing through the axis and the centre of gravity a pressure equal to $\frac{W}{g}x\omega^2$ acting perpendicularly to the axis through a certain point, whose position depends on the form of the body;

(c) in a plane passing through the axis and perpendicular to the former, a pair of equal parallel pressures acting towards contrary parts constituting a couple (Art. 39.), whose moment depends on the angular velocity and the form of the body.

In most other cases the pressures on the axis vary from instant to instant, and are of a much more complicated character than those mentioned above.

CHAP. VII.

ON THE WORK ACCUMULATED IN A BODY THAT ROTATES ON A FIXED AXIS.

135. *The work accumulated in a moving body.*—If all the pressures that act on a body are considered, viz. both those which tend to accelerate and those which tend to retard its motion, it will be evident that the number of units of work accumulated in a given interval is the excess of the number of units done by the former over those done by the latter; in other words, it is the (algebraical) sum of the units of work done by the impressed pressures; let this be denoted by the letter U. Now it will be remembered that the effective pressures at any instant applied in opposite directions would be in equilibrium with the impressed pressures, (Art. 133) and consequently (Art. 90) the sum of the units of work done by the impressed pressures will equal the sum of the units of work done by the effective pressures. Let now the different points of which the body is made up be considered, let their masses be severally $\frac{w_1}{g}, \frac{w_2}{g}, \frac{w_3}{g}, \ldots$ and at the beginning of the given interval let their velocities be severally $V_1, V_2, V_3 \ldots$ and at the end of it $v_1, v_2, v_3 \ldots$ then (Ex. 644) if they had moved separately the number f units of work done upon them respectively would have been $\frac{w_1}{2g}(v_1^2 - V_1^2)$, $\frac{w_2}{2g}(v_2^2 - V_2^2)$, $\frac{w_3}{2g}(v_3^2 - V_3^2)$, ..

Now these must be the works done by the effective pressures, and therefore

$$U=\frac{w_1}{2g}(v_1^2-V_1^2)+\frac{w_2}{2g}(v_2^2-V_2^2)+\frac{w_3}{2g}(v_3^2-V_3^2)+\ldots.$$

Proposition 32.

If a body moves round a fixed axis, and in a given interval its angular velocity is changed from Ω to ω, then the algebraical sum of the number of units of work done upon it during that interval, equals $\frac{1}{2}$ $I_1(\omega^2-\Omega^2)$, where I_1 is the moment of inertia of the body with reference to the axis.

For conceive the body to be made up of heavy points whose respective weights are $w_1, w_2, w_3 \ldots.$ and their perpendicular distances from the axis $r_1, r_2, r_3 \ldots.$, then using the notation of the last article, we have

$$V_1=r_1\,\Omega_1,\ V_2=r_2\,\Omega,\ V_3=r_3\,\Omega\ldots.$$
$$\text{and } v_1=r_1\,\omega,\quad v_2=r_2\,\omega,\quad v_3=r_3\,\omega\ldots.$$

therefore the number of units of work done upon it during the interval equals

$$\frac{w_1\,r_1^2}{2g}(\omega^2-\Omega^2)+\frac{w_2\,r_2^2}{2g}(\omega^2-\Omega^2)+\frac{w_3\,r_3^2}{2g}(\omega^2-\Omega^2)+\ldots$$

which equals $\frac{\omega^2-\Omega^2}{2g}\,(w_1\,r_1^2+w_2\,r_2^2+w_3\,r_3^2+\ldots)$;

now $\frac{1}{g}\,(w_1\,r_1^2+w_2\,r_2^2+w_3\,r_3^2+\ldots)$ is the moment of inertia (I_1) with respect to the axis of rotation; consequently the number of units of work done upon the body equals $\frac{\omega^2-\Omega^2}{2}I_1$.

Q. E. D.

Cor.—If the body begins to move from rest, the number of units of work done on the body equals $\frac{\omega^2\,I_1}{2}$. Now if we consider the axis to be a line, and the body to move

under its own weight, the only pressures acting on it are its weight W, and the reaction of the axis; but since the point of application of the latter force does not move, it does no work; and if the centre of gravity falls through a height h, the former does $\mathrm{W}h$ units of work; therefore the angular velocity acquired by the body under these circumstances is given by the equation

$$\frac{\omega^2 \mathrm{I}_1}{2} = \mathrm{W}h$$

or

$$\omega^2 = \frac{2\,\mathrm{W}h}{\mathrm{I}_1} = \frac{2gh}{k_1^2}$$

where k_1 is the radius of gyration, with reference to the axis of rotation.

Ex. 717.—A rod of cast iron 3 ft. long, $\frac{3}{4}$ of an inch wide, and $1\frac{1}{2}$ inches deep, turns round one of its shortest edges from an angle of 45° with the horizon, find the angular velocity it has when in a horizontal position—its moment of inertia being reckoned that of a *rod*. *Ans.* 4·757.

[See Example 681.]

Ex. 718.—In the last Example determine (1) the velocity in feet per second with which the end of the rod moves, and (2) the number of degrees through which the rod would move in one second if it continued to move uniformly with the angular velocity acquired.

Ans. (1) 14·271. (2) 272° 33′.

Ex. 719.—A cone turns round a horizontal spindle, passing through its vertex at right angles to its axis, what angular velocity will it acquire in falling from its highest to its lowest position? *Ans.* $\omega^2 = \frac{20hg}{4h^2 + r^2}$.

[See Example 694.]

Ex. 720.—In the last Example if the cone is of brass, and is 4 ft. high and its base 1 foot in radius, what pressure will be produced on the axis by its centrifugal force when in its lowest position? and how many times greater than the weight is this pressure? *Ans.* (1) 8116 lbs. (2) $3\frac{9}{13}$ times.

Ex. 721.—If the mass of cast iron described in Example 12 move round the axis assigned in Example 696, determine (1) the angular velocity it acquires in falling from an inclination of 30° to a horizontal position, and (2) the number of units of work accumulated in it.

Ans. (1) 2·21. (2) 5638 units.

Ex. 722.—A cone of cast iron 16 in. high the radius of whose base is 8 in. is fastened to the end of a shaft 4 ft. long and whose end coincides with its centre of gravity at right angles to its axis, the whole moves about a hori-

zontal axis at right angles to the shaft and passing through its extremity; the centre of gravity of the cone descends through a vertical height of 2ft., find the angular velocity acquired. [See Ex. 697.] *Ans.* 2·817.

Ex. 723.—If the oak door described in Example 17 is pushed open by a pressure of 5 lbs. acting at every instant perpendicularly to its face and at a distance of two feet from the inner edge of the door; determine the angular velocity acquired in moving through an angle of 90°. *Ans.* 1·476.

[The number of units of work done on the door is, of course, 5π, so that $\omega^2 I = 10\pi$. See Ex. 698.]

Ex. 724.—A pulley whose moment of inertia is Wk^2 and radius r turns freely round a horizontal axis, a fine thread is wrapped round it to the end of which a weight W_1 is tied; the weight of the string and the passive resistances being neglected, show that if ω is the angular velocity of the pulley when W_1 has descended through h feet, then

$$\omega^2 = \frac{2 W_1 g h}{W_1 r^2 + W k^2}.$$

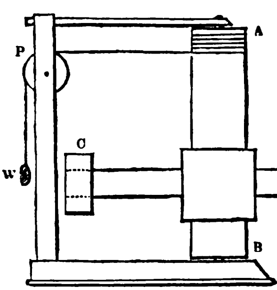
[It must be remembered that when the angular velocity of the pulley is ω the velocity of W_1 is $r\omega$.]

Ex. 725.—A cylinder with its axis vertical turns round a fine spindle coinciding with its axis; a thread is wrapped round the cylinder and then passes horizontally over a pulley capable of revolving round a horizontal axis; to the end of the thread is tied a weight W_1; if $\frac{wk^2}{g}$, $\frac{WK^2}{g}$, are the moments of inertia of the pulley and cylinder, and r and R their radii, and ω the angular velocity of the cylinder after W_1 has fallen through a height h; show that if the passive resistances are neglected

$$\omega^2 = \frac{2gh}{R^2} \times \frac{W_1}{W_1 + W\frac{K^2}{R^2} + w\frac{k^2}{r^2}}.$$

136. *Smeaton's machine.*—For the purpose of testing the truth of the formula for the angular velocity, and consequently of the principles from which that formula is deduced, a machine was invented by Smeaton, which may be described as follows: AB is a cylinder capable of revolving round a very fine and smooth vertical spindle coinciding with its axis; it is crossed at right

Fig. 154.



angles by an arm CD, whose axis is bisected by that of AB, on which are two masses of lead of a hollow cylindrical form, and capable of being shifted backward and forward on their respective arms. The whole is set in motion by a weight W attached to the end of a string, which, after passing horizontally over a small pulley P, is wrapped round the cylinder AB.

Ex. 726.—In Smeaton's machine given the following dimensions, AB is 3ft. 8in. long, and 6in. in diameter, CD is 4ft. long and 3in. in diameter, they are joined by a centre of oak, in shape a cube 8in. along the edge, the masses of lead are 6in. in external diameter and 3in. long; the string is long enough to cause the machine to make 15 turns before it is unwound; determine the angular velocity communicated to the machine by a weight of 20lbs. (1) when the leaden cylinders are placed at the ends of the arms; (2) when they touch the faces of the cube:—the inertia of the pulley, the weight of the string and the passive resistances being neglected.

Ans. (1) 17·3. (2) 49·413.

[Employing the results obtained in Example 701 it is easily shown that the moment of inertia of the revolving piece is 6·33 in the first case, and 0·933 in the second case.]

Ex. 727.—In the first case of the last Example determine approximately the error in the angular velocity that results from omitting the inertia of the pulley, supposing it to be of brass, and to be 2in. in radius and $\frac{1}{2}$ an inch thick.

Ans. 0·0023.

Ex. 728.—There is a pulley whose radius is r, and radius of axle ρ, the limiting angle of resistance between the axle and its bearings is ϕ; a rope (whose weight is to be neglected) is wrapped round this pulley and carries at its end a weight P; given W the weight of the pulley and $\frac{Wk^2}{g}$ its moment of inertia, determine the angular velocity acquired by the pulley when P has fallen through h feet.

[It must be remembered that if the wheel were to move with a uniform motion the number of units of work done upon it would equal $\frac{hW\rho \sin \phi}{r-\rho \sin \phi}$ therefore the number of units of work accumulated equals $h\left\{P-\frac{W\rho \sin \phi}{r-\rho \sin \phi}\right\}$ whence we obtain

$$\omega^2=\frac{2gh\{Pr-(P+W)\rho \sin \phi\}}{(Pr^2+Wk^2)(r-\rho \sin \phi)}.\Bigg]$$

Ex. 729.—A cylinder turns round an axle whose radius is ρ, it starts with

an angular velocity ω, show that it will be brought to rest by friction after n turns, where

$$n = \frac{r^2 \omega^2}{8 \pi \rho g \mu}.$$

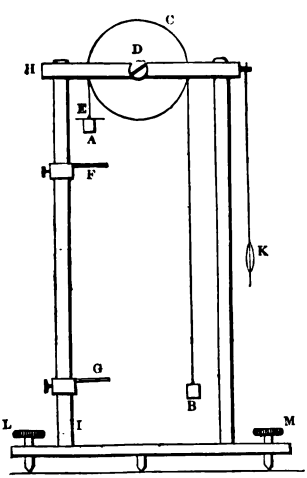
Ex. 730.—The grindstone described in Ex. 16 turns on a bearing of cast iron; it makes 15 turns per minute; determine the number of turns it will make when left to itself, the axle being well greased ($\mu = 0{\cdot}075$, see p. 127). *Ans.* 1·94.

[The moment of inertia may be taken as equal to that found in Ex. 702 and the weight to that found in Ex. 16.]

Ex. 731.—Round the wheel described in Ex. 703 is wound a rope 30 ft. long to the end of which is attached a weight of 250 lbs.; the coefficient of friction between the axle and its bearing is 0·075; the weight is allowed to run down; determine the number of revolutions made by the wheel after the rope has run out, supposing that the rope does not slide on the surface of the wheel during any part of the motion. *Ans.* 5·88 times.

137. *Atwood's machine*, was invented for the purpose of determining the accelerating force of gravity; for practical purposes this can be far more accurately done by means of observations on the pendulum; it however presents a case of terrestrial motion which admits of very accurate observation, and thus supplies a means of testing the truth of the fundamental principles of dynamics: the annexed figure represents an elevation of this machine, which can be sufficiently described as follows. A and B are boxes containing equal weights, and connected by a thread ACB passing over a pulley C, which is supported either on friction wheels or by the points of screws, one of which is seen at D. The box A is made to descend either

Fig. 155.



by a flat weight placed on it or by a bar E, which is intercepted by the ring F, through which the box passes and continues to descend till it strikes the stage G; the space passed over is measured by a scale on H I, and the time by a pendulum K, which may be kept in motion by a clock escapement with a weight: the machine is levelled by the screws L, M.* The weight E produces a certain velocity while moving over a given space, viz. till E comes to F; the velocity acquired is then determined by observing the time in which A moves from F to G; for when E is removed, the boxes A and B will of course move uniformly with the velocity acquired.

Ex. 732.—In Atwood's machine if W is the weight A or B, and P the weight of the bar, and if $\frac{w}{g}k^2$ is the moment of inertia of the pulley and r its radius, then V, the velocity acquired by the machine while P moves through a space h, is given by the formula

$$V^2 = \frac{2gh\,Pr^2}{(2W+P)\,r^2 + wk^2}.$$

[In this result the weight of the thread and the passive resistances are neglected; consequently in comparing it with experiment great care must be taken to suspend the axis of the pulley; and a very fine strong thread should be employed.]

Ex. 733.—If in Atwood's machine the pulley were a solid cylinder of cast iron 2 ft. in diameter, and 3 in. thick, the equal weights 28 lbs. each, the bar 2 lbs., what velocity will the weights have acquired when the preponderating weight has fallen through 15 ft.? *Ans.* 2·853 ft. per sec.

[It may be observed that in the ordinary form of Atwood's machine the wheels are light brass wheels—not at all resembling that described in the Example.]

138. *The Flywheel.*—When a steam engine is employed as a prime mover, it is desirable that the angular velocity communicated to the principal shaft should be as nearly as possible uniform; now it commonly happens that the driving pressure is variable, or else acts at a variable distance (as in the case of a crank); it may also happen that the work to be done by the shaft is intermittent; for instance, it

* Young's Lectures.

may be required to lift a tilt hammer. Now if a sufficiently large flywheel is made to turn with the shaft there will be accumulated in it a number of units of work very much greater than that done by a single turn of the crank, or than the number expended on a single lift of the hammer, and consequently the variations produced in the angular velocity will be very small — the diminution of these variations being the end to be attained by the flywheel. In the examples that follow, it is supposed that the weight of the wheel (W) is distributed uniformly along the circumference of the circle described by the mean radius (r). The moment of inertia of the wheel is therefore $\frac{Wr^2}{g}$. A more accurate determination of the moment of inertia could be obtained as in Ex. 703.

Ex. 734.—An engine of 35 horse-power makes 20 revolutions (*i.e.* up and down strokes) per minute, the diameter of the flywheel is 20 ft., and its weight 20 tons, determine the number of units of work accumulated in it; and if the work done during half a revolution were lost, determine what part of the angular velocity would be lost by the flywheel.

Ans. (1) 307,000 units. (2) $\frac{\omega}{21}$.

Ex. 735.—If the engine in the last Example were employed to lift a tilt hammer weighing 4000 lbs. the centre of gravity of which is raised 3 ft. at each stroke, and if this were done once merely by the work accumulated in the flywheel, what part of its angular velocity would it lose?

Ans. $\frac{\omega}{51}$.

Ex. 736.—If the axis of the flywheel were 6 in. in diameter, and ($\mu =$ 0·075) were of wrought iron turning on cast iron well greased, determine approximately the fractional part of the 35 horse-power expended on turning the flywheel for one minute. *Ans.* $\frac{1}{11}$th.

Ex. 737.—If the flywheel in Ex. 734 were divided into two pieces along a diameter, and if each piece were connected with the axle by a spoke at right angles to that diameter, determine the strain on each spoke arising from centrifugal force; if the velocity of the wheel were liable to be raised to 40 turns per minute, what ought to be the section of a wrought iron spoke which would bear this strain *with safety?*

Ans. (1) 19548 lbs. (2) 11·5 sq. in.

[See Ex. 237 and Art. 9.]

139. *M. Morin's experiments on friction.*—A full account of M. Morin's experiments will be found in his "Notions Fondamentales," already frequently referred to; it would be inconsistent with the plan of the present work to enter into the details of the methods he employed; it may, however, be stated that the arrangement adopted was in principle the same as that described in Ex. 569; to which it must be added that the rope supporting P was of considerable thickness, and passed over a pulley on the edge of the table. Now it will be remarked that in Ex. 569 and 576, it is implicitly assumed that the tension on the horizontal portion of the rope is equal to the tension on the vertical portion; but as in the present case the rope is thick, the axle of the pulley rough, and work is expended in overcoming the inertia of the pulley, this assumption is untrue, and the formulæ given in those examples are inapplicable; the formulæ actually employed will be seen in the following questions; the student will probably find little difficulty in investigating them. The notation adopted is as follows: P denotes the weight producing motion, T the tension on the horizontal portion of the rope; w the weight of the pulley, I its moment of inertia, r its radius, r_1 the radius of its axle, μ the coefficient of friction between the axle and its bearing, a the coefficient of the rigidity of the rope, so that $(1+a)$ T is the pressure to be overcome by P in its descent, f the acceleration of P's motion, g the accelerating force of gravity. The acceleration produced by the weight of the rope is neglected. The mode of determining f will be understood from the next question.

Ex. 738.—If a drum revolves in such a manner that a point on its circumference receives a uniform acceleration f, and if a sheet of paper is wrapped on it, and a pencil with its point resting on the paper is made to move in a direction parallel to the axis with a uniform velocity of V feet per second, show that the curve described on the paper will be a portion of

a parabola, and that if C is the semi-latus rectum measured in feet, we shall have $f = \frac{V}{C}$.

[In the experiments the parabolic curve was unmistakeably obtained, whence immediately follows the important law that friction is independent of velocity.]

Ex. 739.—In M. Morin's experiments show that the pressure between the axis of the pulley and its bearings is given by the formula

$$\sqrt{\left(P + w - P\frac{f}{g}\right)^2 + T^2} \text{ or } 0{\cdot}96\,P\left(1 - \frac{f}{g}\right) + 0{\cdot}96\,w + 0{\cdot}4\,T.^*$$

Ex. 740.—The second formula in the last Example being employed, show that T is given by the formula

$$T\left(1 + a + 0{\cdot}4\frac{\mu r_1}{r}\right) = P\left(1 - \frac{f}{g}\right)\left(1 - 0{\cdot}96\frac{\mu r_1}{r}\right) - 0{\cdot}96\frac{\mu w r_1}{r} - \frac{If}{r^2}.$$

Ex. 741.—A body whose weight is W is caused to slide on a rough horizontal plane by a pressure T, after moving through s ft. it acquires a velocity v, show that the coefficient of friction (μ) is given by the equation

$$\mu = \frac{T}{W} - \frac{v^2}{2gs}.$$

140. *Compound pendulums.*—The terms centre of suspension and centre of oscillation have already been explained (Art. 115); their properties are proved in the following propositions.

Proposition 33.

If k_1 is the radius of gyration of a body with reference to its axis of suspension, and h the distance of the centre of gravity below the centre of suspension, then $\frac{k_1^2}{h}$ is the distance of the centre of oscillation from the latter point.

* The theorem that $\sqrt{a^2 + b^2} = 0{\cdot}96\,a + 0{\cdot}4\,b$ where $a > b$, with an error not exceeding $\frac{1}{25}$th part of the true value, is due to M. Poncelet; it may be proved as follows:—let $a = r \sin\theta$, $b = r\cos\theta$ $\therefore$ $r = \sqrt{a^2 + b^2}$, and θ must have some value between 45° and 90°. Now if $r' = 0{\cdot}96\,a + 0{\cdot}4\,b$ we have $r' = r\,(0{\cdot}96 \sin\theta + 0{\cdot}4\cos\theta)$ but $0{\cdot}4 = 0{\cdot}96 \tan 22° \, 30'$, therefore $r' = r \times 0{\cdot}96\,\frac{\sin(\theta + 22° \, 30')}{\cos 22° \, 30'}$. Then, as θ increases from 45° up to 67° 30′, r

Let AB be the body oscillating about an axis passing through S perpendicularly to the plane of the paper, which also contains the centre of gravity G; join SG, draw the vertical line SC, let G_1 be the position of G at the commencement of the motion, draw G_1M_1 and GM at right angles to SC, and denote G_1SC and GSC by θ_1 and θ respectively. Now the work done by the weight of the body in falling from G_1 to G equals $W = M_1M$, *i.e.* $Wh(\cos\theta - \cos\theta_1)$, and therefore, if ω is the angular velocity acquired, we have (Prop. 32)

Fig. 156.

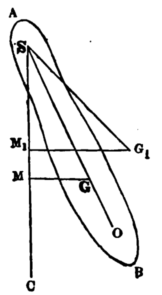
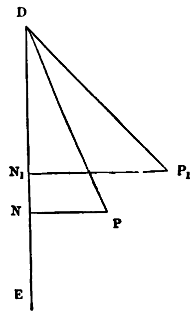


Fig. 157.



$$\frac{1}{2}\frac{W}{g}\omega^2 k_1^2 = Wh(\cos\theta - \cos\theta_1)$$

$$\therefore \omega^2 = \frac{2gh}{2k_1^2}(\cos\theta - \cos\theta_1)$$

Let DP be a simple pendulum oscillating about D, draw the vertical line DE, and let P_1 be the position from which P begins to move; draw PN and P_1N_1 at right angles to DE, and let DP be denoted by l, and let P_1DE equal θ_1, and PDE equal θ; then if v is the velocity acquired by the point in falling from P_1 to P, we have

$$v^2 = 2g \times NN_1 = 2gl(\cos\theta - \cos\theta_1)$$

will increase from $0{\cdot}96r$ to $1{\cdot}04r$, and as θ increases from 67° 30′ up to 90°, r' decreases from $1{\cdot}04r$ to $0{\cdot}96r$, and consequently r' never differs from r by more than $\frac{r}{25}$.

and therefore, if ω' is the angular velocity of P, we have

$$\omega'^2 = \frac{2g}{l}(\cos\theta - \cos\theta_1).$$

Now if l equals $\frac{k_1^2}{h}$, ω' will equal ω for all values of θ, and since AG and DP are moving at each instant with the same angular velocity, their oscillations will be performed in the same time, and therefore $\frac{k_1^2}{h}$ is the length of the simple pendulum oscillating in the same time as AB, hence if in SG produced a point O be taken, such that SO equals $\frac{k_1^2}{h}$, that point will be the centre of oscillation.

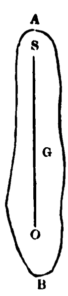
Cor.—The time of a small oscillation of AB will equal $\frac{\pi k_1}{\sqrt{gh}}$.

Proposition 34.

The centres of oscillation and suspension are reciprocal.

Let AB be the body, G its centre of gravity, S a centre of suspension, and O the corresponding centre of oscillation, it is to be proved that these points are reciprocal, *i. e.* if O is made the centre of suspension S will be the corresponding centre of oscillation. Let k be the radius of gyration round a parallel axis through the centre of gravity, let SG, GO be respectively denoted by h and x,

Fig. 158.



$$\therefore\ x + h = \frac{k^2 + h^2}{h}$$

$$\text{or}\quad hx = k^2.$$

Next let O be the centre of suspension, and y the

distance from G to the corresponding centre of oscillation, then

$$y+x=\frac{k^2+x^2}{x}$$

$$\text{or}\quad yx=k^2$$

and therefore y equals h, or S is the centre of oscillation.

Q. E. D.

Ex. 742.—A thin rod of steel 10 ft. long vibrates about an axis passing through one end of it; determine the time of a small oscillation; the number of vibrations it makes in a day; and the number it will lose in a day if the temperature is increased by 15° F.

Ans. (1) 1·434 sec. (2) 60254. (3) 3.

Ex. 743.—A pendulum vibrates about an axis passing through its end; it consists of a steel rod 60 in. long, with a rectangular section $\frac{1}{2}$ by $\frac{1}{4}$ of an inch; on this rod is a steel cylinder 2 in. in diameter and 4 in. long; when the ends of the rod and cylinder are set square determine the time of a small oscillation. *Ans.* 1·174.

Ex. 744.—Determine the radius of gyration with reference to the axis of suspension of a body that makes 73 oscillations in 2 minutes, the distance of the centre of gravity from the axis being 3 ft. 2 in. *Ans.* 5·267 ft.

Ex. 745.—Determine the distance between the centres of suspension and oscillation of a body that vibrates in $2\frac{1}{2}$ sec. *Ans.* 20·264 ft.

Ex. 746.—If $\frac{k_1^2}{h}$ is the length of a simple pendulum corresponding to a vibrating rod; show that if it expands uniformly in the proportion of $1+\alpha : 1$ that the length of the simple pendulum becomes $(1+\alpha)\frac{k_1^2}{h}$.

Ex. 747.—Miaran determined the length of the seconds pendulum at Paris to be 39·128 inches; he employed a ball of lead 0·533 inches in diameter suspended by an exceedingly fine fibre whose weight could be neglected *; supposing the measurements made with perfect accuracy, upon the supposition that the distance from the point of suspension to the centre of the ball is the length of the pendulum, show that the error is less than the 0·001 of an inch.

Ex. 748.—A pendulum consists of a brass sphere 4 in. in diameter suspended by a steel wire $\frac{1}{10}$th of an inch in diameter; the centre of the sphere is 40 inches below the point of support †; determine the number of oscillations it will make in a day; and what number would be obtained on the suppo-

* Airy, Figure of the Earth, p. 224. † Do. p. 225.

sition that the centre of oscillation coincides with the centre of the sphere ($g=32$). *Ans.* (1) 85243. (2) 85212·7.

Ex. 749.—If a sphere whose radius is r is suspended successively from two points by a very fine thread, and if the distances of the centre of the sphere from the points of suspension are respectively h and h', and if l and l' are the distances of the corresponding centres of oscillation from the points of suspension, show that

$$l-l'=(h-h')\left(1-\frac{2r^2}{5hh'}\right).$$

Ex. 750.—If t and t' are the times of a small oscillation of the pendulum in the last Example corresponding respectively to l and l'; show that the accelerating force of gravity is given by the equation

$$g=\frac{\pi^2\,(h-h')}{t^2-t'^2}\left(1-\frac{2r^2}{5hh'}\right).$$

141. *M. Bessel's determination of the accelerating force of gravity.*—The last two examples contain the principle of the method by which M. Bessel determined the accelerating force of gravity at Königsberg.* The pendulum was first allowed to swing from a point of support at a distance h above the centre of the sphere, and the number of oscillations made in a given time were noted, by which t was determined with great accuracy; the wire was then grasped between two points that were screwed together, and now the oscillations were performed about a point distant h' from the centre of the sphere, and t' noted as before; now $h-h'$ being the distance between two fixed points, admits of very accurate determination; the lengths h and h' cannot be determined without some liability to error, but as they only appear in the small term $\frac{2r^2}{5hh'}$, that error will hardly affect the determination of g, which can by this method be ascertained with extreme accuracy.

Ex. 751.—In the last Example let r, h, and h' be respectively reckoned 1, 50, and 40 inches, so that $h-h'$ is exactly 10 inches, but it is doubtful whether the separate values of h and h' are not as much as $\frac{1}{10}$th of an inch

* Airy, Figure of the Earth, p. 223.

longer than the values assigned, determine the possible error in the value of g. *Ans.* $\frac{g}{1116000}$.

142. *Captain Kater's method of determining the accelerating force of gravity.*—This method depends on the reciprocity of the centres of oscillation and suspension; the pendulum has two axes (or knife edges as they are called, though they are really wedges of very hard steel), by either of which it can be suspended; now if the time of oscillation about either axis is the same, the distance between the edges will be the length of the simple pendulum, and the distance being that between two fixed points, admits of very accurate measurement, and then g is obtained by the formula

$$g=\frac{\pi^2}{t^2}.l.$$

The difficulty of giving the edges their exact position is overcome as follows: on the pendulum rod is placed a weight that can be moved up or down by screws; the edges are fixed as nearly as possible in the right position; and then by moving the weight up or down, the values of k_1 and h can be changed until $\frac{k_1^2}{h}$ equals the distance between the edges, *i. e.* until the number of oscillations made in a given time about either edge is the same.

143. *The motion of a body through space.*—A body *may* so move that its different points describe in space a system of parallel straight lines; when this is the case, the body is said to have a *motion of translation,* and its velocity will of course be the same as that of any one of its points. *In general* the motion of a body is by no means so simple, each of its points moving in a particular curve (Art. 131); it is capable of proof, however, that the motion can at each instant be represented by the co-

existence of two motions, (1) a translatory motion whose velocity and direction is the same as that of the centre of gravity *; (2) a rotatory motion round an axis passing through the centre of gravity. Now it must be remembered that from instant to instant the velocity and direction of the translatory motion will *in general* change, and the velocity of the rotatory motion will also *in general* change as well as the axis about which the rotation takes place; it is evident from this that the general discussion of the motion of a rigid body is one that presents great difficulties.

Now it will be remarked that if we suppose the motion to take place in one plane †, the rotation must take place about an axis perpendicular to that plane; this introduces a great simplification, and as it is desirable that the reader should have an opportunity of considering this general case of motion, though our limits will not allow of its complete discussion, we will prove in regard to the above limited case:—(1) That the motion of the body can be correctly represented by coexistent motions of translation and rotation; (2) A formula for determining the number of units of work accumulated at any instant in such a body.

It may be assumed as evident, that if a system of points is in motion and a common velocity is impressed on them all, there will be no change in their relative motions. To illustrate this, suppose a number of persons dancing on board a ship in a state of steady motion, their movements are evidently the same whether the velocity of the ship be greater or smaller; and consequently those movements would be unaffected by a change in the ship's velocity

* This fact might be ennunciated with reference to any other point: nor will our investigation proceed sufficiently far to show the reason for fixing on the centre of gravity.

† *i.e.* when one particular section of the body is always in the same plane so that each parallel section always moves in a parallel plane.

after the change had occurred. During the change, *i. e.* before they had themselves acquired the same velocity as the vessel, they might feel a jerk, but afterwards their movements would continue as before. Now let v be the velocity of the centre of gravity, impress on each point of the body a velocity equal and opposite to v; this will not affect the motion of the points relative to the centre of gravity, and since that is brought to rest, the relative motion will be all that is left, and will be a motion of rotation round the centre of gravity; now the whole motion of the body must have been the part destroyed together with the part remaining, *i. e.* it must have consisted of a motion of translation whose velocity is the same and in the same direction as that of the centre of gravity, and a motion of rotation round the centre of gravity.

Again let $w_1, w_2, w_3 \ldots.$ be the weights of the different points composing the body, the whole weight of which may be denoted by W, also let k be the radius of gyration, and ω the angular velocity of the body round the axis passing through the centre of gravity. Now the total number of units of work destroyed when the velocity v was impressed on each point must have been

$$\frac{w_1}{2g}v^2 + \frac{w_2}{2g}v^2 + \frac{w_3}{2g}v^2 + \ldots \text{ or } \frac{W}{2g}v^2$$

and the number of units of work left in the body equals $\frac{Wk^2}{2g}\omega^2$; so that the number actually accumulated must have been

$$\frac{W}{2g}v^2 + \frac{Wk^2}{2g}\omega^2$$

Ex. 752.—A wheel whose diameter is 4 ft. moves with a true rolling motion, it makes 50 turns a minute, determine the velocities of the highest point and of the two extremities of the horizontal diameter.

Ans. **(1) 20·94. (2) 14·81.**

Ex. 753.—A circular plate is rigidly attached to an axis which passes at right angles to its plane through the middle point of a radius, the axis makes 30 uniform revolutions per minute; determine the velocity of translation of the centre of gravity and of rotation round the centre of gravity by which the motion may at each instant be represented, and to verify the results deduce from them the velocities of the extremities of that diameter which passes through the axis. *Ans.* (1) $\frac{1}{2}\pi r$. (2) π. (3) $\frac{1}{2}\pi r$, $\frac{3}{2}\pi r$.

Ex. 754.—If the moon's velocity in her orbit were uniform and the orbit a perfect circle, and if the moon always presented the same face to the earth, what would be the nature of the moon's motion?

Ex. 755.—If a penny piece were set flat in the plane of a wheel; show that its motion would precisely resemble that of the moon as described in the last article.

Ex. 756.—If the flywheel in Ex. 734 had been rolling along the ground and making the specified number of revolutions, determine the number of units of work accumulated in it. *Ans.* 614000.

Ex. 757.—A sphere whose radius is r rolls from rest down a length l of a plane whose inclination to the horizon is θ, show that if V is the acquired velocity of the centre of gravity then $V^2 = \frac{10}{7} g l \sin \theta$.

[The work accumulated equals $W l \sin \theta$, since the reaction does no work (Art. 92); also if ω is the angular velocity it is plain that $V = r\omega$.]

Ex. 758.—A drum open at both ends whose external radius is r rolls over the length l of the plane in the last Example in t seconds, show that r_1 the internal radius, is given by the equation

$$r^2{}_1 = r^2 \left\{ \frac{g t^2 \sin \theta}{l} - 3 \right\}.$$

Ex. 759.—A string is wrapped round a cylinder whose weight is W and radius r. If the cylinder descends by its own weight determine the *time* in which a length l of the string will be unwound.

[The tension of the string will do no work (Art. 92).]

CHAP. VIII.

ON THE ACTION OF IMPULSIVE FORCE.

144. *Impulsive action.*—Suppose a sphere A to overtake a sphere B, their centres moving in the same line; it is a matter of common observation that they will strike, and then separate, A moving after impact with a less, and B with a greater velocity than before; the problem we are to solve is this:—Given the weights of the bodies and their velocities at the instant before impact, to determine the velocities they have at the instant after impact.

Now it will be observed that though the bodies are in contact during a very short time, yet that time is really finite, and the pressure which the one exerts on the other must increase from zero at the instant of contact, till it attains a very considerable magnitude, and then decreases down to zero at the instant of separation. Moreover, it appears from Ex. 656, that if A exerted at each instant against B, a pressure equal to that which B exerts against A, in other words if the action and reaction were equal and opposite pressures, then the momentum lost by A would equal that gained by B, and the total amount of momentum in A and B before impact would equal the total amount after impact. Now, that this is a fact, was ascertained by numerous experiments make by Newton*, and this we shall take as our fundamental principle, viz. *that the momentum lost during the impact by one body will equal that gained by the other.* For the purpose of completing the above statement, it may be added that the

* Introduction to the Principia.

sum of the momenta of the two bodies means their *algebraical* sum.

145. *The mean pressure exerted during impact.*—The following example is intended to illustrate the fact that there is really called into play a very large pressure which is exerted during a very short time.

Ex. 760.—A hard mass weighing 50 lbs. falls from a height of 6 ft. on a plane surface which at the instant of greatest compression has yielded to the extent of $\frac{1}{80}$th of an inch—the mass itself being supposed to be entirely uncompressed—determine the mean mutual pressure, and the duration of compression.* ***Ans.* (1) 72000 lbs. (2) 0·000425 sec.**

[The pressure must be such that by acting through $\frac{1}{80}$th of an inch brings the mass to rest:—the duration is the time in which the *mean pressure* would bring the body to rest.]

146. *Impact of inelastic bodies.*—When A overtakes B, it is plain that so long as A moves faster than B, the two surfaces of contact will be compressed, and the state of compression will continue until A and B are moving with the same velocity; if the mutual action then ceases, the bodies are said to be inelastic.

Now let the masses of A and B equal $\frac{A}{g}$ and $\frac{B}{g}$ respectively, let R be the momentum lost by the one and gained by the other during impact, and let their velocities before impact be V and U, and their common velocity after impact equal v; then we obtain from the fundamental principle (Art. 144.)

$$\frac{A}{g}v = \frac{A}{g}V - R \quad \ldots\ldots\ldots (1)$$

$$\frac{B}{g}v = \frac{B}{g}U + R \quad \ldots\ldots\ldots (2)$$

whence

$$R = \frac{AB\,(V-U)}{g\,(A+B)} \quad \ldots\ldots (3)$$

$$v = \frac{AV+BU}{A+B} \quad \ldots\ldots (4)$$

*** Poncelet, Introd. à la Méc. Ind. p. 166.**

In working examples the student is recommended to proceed from the general principle, or, in other words, to form and then solve the equations (1) and (2), and not to substitute particular values in (3) and (4). If A *meets* B, one of the velocities must be reckoned negative, and the bodies will move after impact in that direction if v is negative.

Ex. 761.—If A weighing 2 lbs. and moving with a velocity of 20 ft. per second overtakes B weighing 5 lbs. and moving with a velocity of 5 ft. per second, determine the common velocity after impact. *Ans.* $9\frac{2}{7}$ ft. per sec.

Ex. 762.—In the last Example if the bodies had met, determine the common velocity after impact. *Ans.* $2\frac{1}{7}$ ft. per sec. in A's direction.

Ex. 763.—In Art. 146 show that the number of units of work lost during impact equals $\frac{AB(V-U)^2}{2g(A+B)}$.

Ex. 764.—If a shot weighing P lbs. is fired with a velocity V into a mass of wood weighing Q lbs. which is quite free to move, show that the velocity with which the wood begins to move is $\frac{PV}{P+Q}$; and state why this case must be one of inelastic impact.

Ex. 765.—If in the last Example $Q=nP$, show that, in consequence of the impact, n units of work are lost in every $n+1$.

147. *Impact of elastic bodies.*—It commonly happens that the mutual action does not entirely cease with the compression, but when that ends the bodies begin to recover their shapes, and thereby continue to press on each other till the impact terminates. Now let R be the momentum lost by the one body and gained by the other during compression, and R′ that lost and gained during expansion; then the whole momentum lost by the one body and gained by the other will equal $R+R'$. But it is found by experiment that for the same substances R bears to R′ a fixed ratio $1:\lambda$*; therefore $R'=\lambda R$, and $R+R'=(1+\lambda)R$; where λ is a constant quantity depending on the materials of the impinging bodies. In the two extreme cases of inelasticity and perfect elasticity, λ equals

* This follows from Newton's experiments already referred to.

0 and 1 respectively; in other cases λ is a proper fraction, and commonly a small one. We have already seen that if a body whose weight is A moving with a velocity V, overtakes another whose weight is B moving with a velocity U, then the momentum lost by the one and gained by the other at the end of compression equals $\frac{AB\,(V-U)}{g\,(A+B)}$. Hence the total momentum gained and lost will equal $(1+\lambda) \times \frac{AB\,(V-U)}{g\,(A+B)}$. And therefore if v and u are their respective velocities after impact, we shall have

$$\frac{A}{g}v = \frac{A}{g}V - (1+\lambda)R$$

$$\frac{B}{g}u = \frac{B}{g}U + (1+\lambda)R$$

or

$$v = V - \frac{(1+\lambda)\,B\,(V-U)}{A+B}$$

and

$$u = U + \frac{(1+\lambda)\,A\,(V-U)}{A+B}$$

It may be added that the remarks made in Art. 146, relative to the working of examples, are applicable to the case of elastic bodies.

Ex. 766.—Show that v and u are given by the following formulæ

$$v = \frac{AV+BU}{A+B} - \frac{\lambda B\,(V-U)}{A+B}$$

$$u = \frac{AV+BU}{A+B} + \frac{\lambda A\,(V-U)}{A+B}.$$

Ex. 767.—Determine the velocities after impact of a ball (A) weighing 20 lbs. which, moving with a velocity of 100 ft. per second, overtakes a ball (B) weighing 50 lbs. and moving with a velocity of 40 ft. per second, their coefficient of elasticity being $\frac{1}{2}$. *Ans.* A's velocity $35\frac{5}{7}$; B's velocity $65\frac{5}{7}$.

Ex. 768.—In the last case suppose the heavier body (B) to be at rest, determine the velocities after impact.

Ans. A rebounds with a velocity $7\frac{1}{7}$. B moves forward with a velocity $42\frac{6}{7}$.

Ex. 769.—Obtain the velocities after impact in Ex. 767, upon the supposition that the bodies meet.

Ans. A rebounds with a velocity 50, and B with a velocity 20.

Ex. 770.—If there are two perfectly elastic balls A and B of equal weight, and A moving with a velocity V impinges on B at rest, show that A is brought to rest and B takes the velocity V. If there are a number of equal and perfectly elastic balls B, C, D, E, placed in a line, what would be the result of A striking B; the direction of the impact coinciding with the line?

Ex. 771.—If a ball whose weight is A moving with a velocity V meets a ball whose weight is B moving with a velocity U, show that in the case of perfect elasticity the velocities of rebound are given by the following construction:—draw any line AB, divide it in G in the inverse ratio of the weights of A and B, and in C in the ratio of their velocities; on the other side of G measure off GD equal to GC, then A's velocity of rebound : B's velocity of rebound :: AD : BD.*

Ex. 772.—Two balls weighing respectively 12 and 8 lbs. are suspended by threads in such a manner that their centres are 4 ft. below the points of support; when at rest the line joining their centres is horizontal; if the smaller one is raised so as to fall through a quadrant, determine the angle described by the other after impact, if the coefficient of elasticity equals $\frac{2}{3}$.

Ans. 56° 14′.

Ex. 773.—If A and B are the weights of two perfectly elastic balls, if V and U are their velocities before impact and v and u their velocities after impact, show that

$$AV^2 + BU^2 = Av^2 + Bu^2.$$

Ex. 774.—If a ball impinges perpendicularly on a fixed plane with a velocity V, show that the velocity of rebound equals λV.

[It must be remembered that at the end of compression the velocity is entirely destroyed.]

Ex. 775.—If bodies are dropped from equal heights on a fixed horizontal plane, show that their coefficients of elasticity are in the same ratio as the square roots of the heights to which they rebound.

Ex. 776.—A ball is dropped from a height h, show that the whole space it describes before coming to rest equals

$$h\,\frac{1+\lambda^2}{1-\lambda^2}.$$

Ex. 777.—A ball (A) is thrown upward with a velocity of 160 ft. per second, when it has reached a height of 300 ft. it is met by an equal ball (B) which has fallen from a height of 100 ft.; determine the time after the instant of impact in which each will reach the ground, assuming that λ equals unity.

Ans. A after $2\frac{1}{2}$ sec. B after $7\frac{1}{2}$ sec.

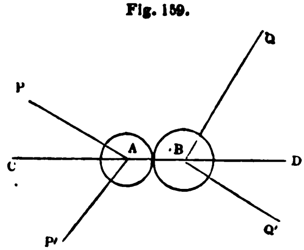
148. *Oblique impact of smooth bodies.*—If a smooth ball impinges obliquely on a smooth plane, or on another

* It was in this form that the problem of impact was originally solved by Sir C. Wren (vide Montucla).

smooth ball, the velocities can be resolved into components in the direction of, and perpendicular to the line of centres; of these, the latter will be unaffected by the impact, the former will be changed in precisely the same manner as if the impact of the bodies had been direct, and they had moved with the component velocities only: the component velocities after impact are thus obtained, and then the whole velocities are found by composition. The general formulæ commonly given for these velocities are of very little value, as any particular example is much more easily worked by proceeding from first principles: the following example will sufficiently exhibit the method of treating these cases.

Ex. 778.—Let A and B be two perfectly elastic balls which at the instant of impact are moving along the lines PA and QB; the line of centres CD being in the same plane as PA and QB. A weighs 10 lbs., moves with a velocity of 16 ft. per second, and the angle PAC contains 30°. B weighs 15 lbs., moves with a velocity of 8 ft. per second, and the angle QBD contains 60°; determine the velocities after impact and their directions.

Fig. 159.



(*a*) Before impact A's velocity at right angles to CD is 8, and B's $4\sqrt{3}$. They are unchanged by the impact.

(*b*) Before impact A's velocity along CD is $8\sqrt{3}$ and B's velocity is -4; these are changed by impact into $-\frac{8}{5}(3+\sqrt{3})$ and $\frac{4}{5}(-1+8\sqrt{3})$ respectively.

(*c*) Hence A's velocity after impact equals $\frac{8}{5}(37+6\sqrt{3})^{\frac{1}{2}}$, and B's velocity $\frac{4}{5}(268-16\sqrt{3})^{\frac{1}{2}}$ *i.e.* A's velocity equals 11·01 ft. per second and B's equals 12·4 ft. per sec.

(*d*) The directions of the motion of P and Q after impact are respectively AP′ and BQ′ where tan P′AC equals $\frac{5}{3+\sqrt{3}}$, and tan Q′BD equals $\frac{5\sqrt{3}}{8\sqrt{3}-1}$ *i.e.* P′AC equals 64° 35′ and Q′BD equals 33° 58′.

By this means the motion of A and B at the instant after impact is completely determined.

Ex. 779.—If a ball A moving in a direction making an angle of 30° with the line of centres overtakes B moving along the line of centres; determine the velocities, if A weighs 12 lbs. and its velocity is 12 ft. per second, and B weighs 30 lbs. and its velocity 4 ft. per second, and the coefficient of elasticity equals $\frac{1}{2}$. *Ans.* (1) A's vel. 7, C A P′ = 149° 30′. (2) B's vel. 6·7.

Ex. 780.—A body whose coefficient of elasticity is $\frac{1}{2}$ impinges with a velocity of 30 ft. per second on a fixed plane in a direction making an angle of 27° with the perpendicular; determine the magnitude and direction of the velocity after impact. *Ans.* (1) 19·1 ft. (2) 45° 32′.

Ex. 781.—If in the last Example the body had been inelastic, how would it begin to move after impact?

Ex. 782.—If in Example 774 the angle of impact is α and the angle of rebound β and the coefficient of elasticity λ, show that

$$\tan \beta = \frac{\tan \alpha}{\lambda}.$$

Ex. 783.—Give a geometrical construction by which to determine the direction in which a billiard ball must begin to move so that after one rebound it may strike another ball whose position is given, (1) if the coefficient of elasticity equals unity, (2) if the coefficient of elasticity equals λ.

Ex. 784.—Extend the construction in the last Example to the case in which the ball makes two rebounds from cushions at right angles to each other.

Remark.—If the surfaces of the impinging bodies are rough, the effect of the tangential impact will generally be to produce a rotatory motion, as well as to modify the previous motion of the bodies: the complete solution of this case lies beyond the scope of the present work. The same remark applies to the case in which the motion of one or both bodies sustains a resistance appreciable in comparison with the mean impulsive pressure.

149. *Application of D'Alembert's principle to the case of impulsive action.*—It will be remarked that a case of impulsive action does not differ essentially from any other case of motion produced by pressure; the difference in the mode of treating these cases arises solely from our inability to determine the pressure exerted at each instant of the duration of the impact; it follows, therefore, that at each instant during the collision, the effective forces applied in

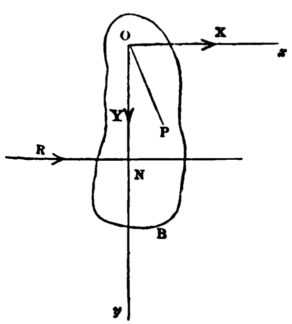
the opposite directions would be in equilibrium with the impressed forces; and consequently the momenta produced by the effective forces so applied, and those actually produced by the impressed forces, will satisfy the conditions of the equilibrium of pressures. We shall apply this principle to determine the angular velocity communicated by a blow to a body capable of revolving round a fixed axis, and the impulse produced on the axis by that blow.

Proposition 35.

A body capable of turning round a given axis, and symmetrical with reference to the plane passing through the centre of gravity at right angles to the axis is struck by a blow of given magnitude along a line lying in that plane, to determine the angular velocity communicated to the body, and the impulse on the axis.

Let the plane of the paper be the plane of symmetry, and let the axis of rotation pass through O; let R be the magnitude of the blow which is delivered along the line RN; draw Oy at right angles, and Ox parallel to RN; let $\frac{W}{g}$ be the mass of the body, $\frac{W}{g}k_1^2$ its moment of inertia with reference to the given axis, $\bar{x}\ \bar{y}$ the co-ordinates of its centre of gravity, and ON equal to a; let X and Y be the impulsive reactions of the axis in the directions of Ox, and Oy respectively; and let ω be the angular velocity of the body communicated by the blow. Consider any particle P whose co-ordinates are x_1, y_1, distance from O equals r_1 and mass $\frac{w_1}{g}$, and let the angle xOP equal θ_1, also suppose

Fig. 160.



a similar notation to be employed for the other particles composing the body. Now the velocity of P is $r_1 \omega$ in a direction perpendicular to OP, or is equivalent to velocities $\omega r_1 \sin \theta$, or ωy_1 parallel to Ox and $-\omega r_1 \cos \theta$, or $-\omega x_1$ parallel to Oy, and therefore the momentum communicated to P is equivalent to the two $\frac{w_1}{g} y_1 \omega$ parallel to Ox, and $-\frac{\omega_1}{g} x_1 \omega$ parallel to Oy; the expressions for all the other particles being precisely similar. But these are the momenta that would be communicated by the effective forces, and the impressed forces are R, Y, and X; also it will be observed that the moment of P's momentum round O is $\frac{w_1}{g} r_1^2 \omega$; consequently (Prop. 13)

$$\mathrm{R} + \mathrm{X} = \frac{w_1}{g} y_1 \omega + \frac{w_2}{g} y_2 \omega + \frac{w_3}{g} y_3 \omega + \ldots$$

$$= \frac{\omega}{g} (w_1 y_1 + w_2 y_2 + w_3 y_3 + \ldots)$$

$$-\mathrm{Y} = \frac{w_1}{g} x_1 \omega + \frac{w_2}{g} x_2 \omega + \frac{w_3}{g} \omega + \ldots$$

$$= \frac{\omega}{g} (w_1 x_1 + w_2 x_2 + w_3 x_3 + \ldots)$$

$$a\mathrm{R} = \frac{w_1}{g} r_1^2 \omega + \frac{w_2}{g} r_2^2 \omega + \frac{w_3}{g} r_3^2 \omega + \ldots$$

$$= \frac{\omega}{g} (w_1 r_1^2 + w_2 r_2^2 + w_3 r_3^2 + \ldots)$$

Or by Prop. 15, and Art. 126.

$$\mathrm{R} + \mathrm{X} = \frac{\mathrm{W}}{g} \bar{y} \omega$$

$$-\mathrm{Y} = \frac{\mathrm{W}}{g} \bar{x} \omega$$

$$a\mathrm{R} = \frac{\mathrm{W}}{g} k_1^2 \omega$$

or $$\omega = \frac{gR}{W} \cdot \frac{a}{k_1^2} \dots\dots\dots\dots (1)$$

$$-Y = \frac{Ra\bar{x}}{k_1^2} \dots\dots\dots\dots (2)$$

$$-X = R - \frac{Ra\bar{y}}{k_1^2} \dots\dots\dots\dots (3)$$

The first of these equations gives the angular velocity communicated to the body; the second and third equations give the components of the reaction of the axis, which is of course equal and opposite to the blow sustained by the axis.

N.B.—It will be an instructive exercise for the student to ascertain for what positions of the centre of gravity the reactions of the axis will be as indicated in the figure: it will commonly happen, as he will find, that the reactions will really act in the contrary directions to those indicated.

Ex. 785.—A uniform rod 12 ft. long and weighing 10 lbs. is suspended at one end, it receives at the other, in a direction perpendicular to its length, a blow whose momentum is 1; determine (1) the angular velocity with which it begins to move, (2) the impulsive pressure on the axis, and (3) compare this impulse with the blow given by a weight of $\frac{1}{4}$ of a pound which has fallen through a height of 2 ft. *Ans.* (1) 0·8. (2) $-\frac{1}{2}$. (3) 4 times.

Ex. 786.—A beam of oak 10 ft. long and 1 ft. square is suspended by an axis perpendicular to one face and passing through the axis of the beam, at a distance of 1 ft. from the end; it is struck at a point 8 ft. below the axis by a bullet weighing 1 lb. and moving with a velocity of 1000 ft. per second; determine (1) the impulse on the axis, (2) the angular velocity communicated to the beam, (3) the angle through which the beam will revolve.

Ans. (1) 10. (2) 0·56. (3) 14° 8′.

Ex. 787.—A hammer's head (considered as a point) weighs 10 lbs. and makes 60 strokes per minute on an anvil, if the time of ascending equals that of descending, and the blow is entirely due to the velocity it acquires in falling, compare that blow with the impulse on the axis in the last Example. *Ans.* One half.

Ex. 788.—Determine the impulse on the axis if the mass of cast iron in Ex. 721 strikes an anvil after falling through the 30°; the blow on the anvil being supposed to be given by the extreme edge of the cube.

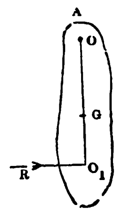
Ans. 97.

[It will be observed that in this case the impulse on the axis is greater

than that which would be produced by a shot weighing 3 lbs. and moving at the rate of 1000 ft. per second; it is obvious that a succession of such impulses would tear to pieces the masonry on which the axis of such a hammer is supported; and accordingly it becomes a point of great practical importance to suspend a tilt hammer in such a manner that there shall be no impulse on the axis. The following explains the principle on which this is done.]

150. *The centre of percussion.*—Referring to the equations (2) and (3) of Prop. 35, we see that if the blow is delivered in such a manner that $\bar{x}$ equals zero, and k_1^2 equals $a\bar{y}$, then X and Y equal zero separately, and there is no impulsive pressure on the axis of suspension; hence if O be the centre of suspension, G the centre of gravity of the body, and a point O_1 be taken in OG produced so that

Fig. 161.



$$O\,O_1 = \frac{k_1^2}{O\,G}$$

then if the body is struck by a blow whose direction passes through O_1 at right angles to OO_1, there will be no impulsive pressure on the axis, and the point O_1 is therefore called the centre of percussion; it evidently coincides with the centre of oscillation with respect to the centre of suspension O. It must be remembered that the body is supposed to be symmetrical with regard to the plane of the paper, as specified in the enunciation of Prop. 35.

Axis of spontaneous rotation.—Since the body in the last article when struck begins to rotate round the axis through O without any constraint, it follows that if the body were entirely free, it would begin to move round that axis, which is therefore called the axis of spontaneous rotation. If it is given that a body is struck by a blow R along a given line, the axis of spontaneous rotation is determined as follows: consider the plane passing through G the centre of gravity and the direction of the blow; through G draw a line at right angles to this plane, and

let k be the radius of gyration of the body with respect to it: through the centre of gravity draw a line at right angles to the direction of the blow and cutting it in O_1, and on the other side of the centre of gravity take in the line a point O such that

$$OG \cdot GO_1 = k^2$$

then an axis through O perpendicular to the given plane is the axis of spontaneous rotation.

It will be observed that if the axis of spontaneous rotation is to pass through the centre of gravity, we must have in equation (3) of Prop. 35, both $\bar{x}=0$ and $\bar{y}=0$, and therefore $R=0$: but from equation (1) ω having a finite value aR must also have a finite value; or in other words the body must be struck by an impulsive couple whose moment is aR, and whose plane passes through the centre of gravity of the body; it will then begin to revolve with an angular velocity $\frac{gaR}{Wk^2}$ round an axis at right angles to the plane of the couple, and passing through the centre of gravity. The remark made in the last article with reference to the symmetry of the body is true in regard to the present article.

Ex. 789.—A hammer turns round a given axis, the weight of the head is W, and its radius of gyration is k with respect to an axis parallel to the given axis and passing through its centre of gravity, the weight of the handle is W_1; its radius of gyration with respect to the axis is k_1, and the distance of its centre of gravity from the axis a. If the head of the hammer is so placed that its centre of gravity is at the same distance (x) from the axis as the centre of percussion of whole hammer, then

$$x = \frac{W_1 k_1^2 + W k^2}{W_1 a}.$$

Ex. 790.—If the head of the hammer in Ex. 721 is shifted so as to fulfil the conditions of the last Example, determine the distance of its centre of gravity from the axis of rotation. *Ans.* 5·35 ft.

Ex. 791.—A sledge hammer AC is moveable round an axis through A; it is 6 ft. long and weighs 4 cwts., it is held in a horizontal position by a weight of 3 cwts. attached to the end of a string which after passing over a

small pulley is fastened to B (the parts of the string being vertical); the hammer when allowed to fall into a vertical position makes 50 oscillations per minute round A; determine (1) the centre of percussion, and (2) the radius of gyration about an axis parallel to the axis of suspension and passing through its centre of gravity. *Ans.* (1) 4·67 ft. (2) 0·87 ft.

Ex. 792.—A cylindrical bolt of cast iron 4 in. in diameter and 8 in. long is struck simultaneously by two equal blows in contrary directions, each at right angles to an extremity of a diameter of its mean section; in consequence the bolt rotates 250 times in a second; determine the magnitude of each blow, and compare it with that which the bolt itself would give if moving with a velocity of 1000 ft. per second. *Ans.* (1) 53·6. (2) $\frac{\pi}{48}$.

151. *Robin's Ballistic Pendulum.*—This machine is employed to ascertain the velocity with which a shot leaves the mouth of a cannon. The principle on which it is constructed will be most easily understood by describing it in its original form; at present the gun itself is suspended and the recoil observed; but at first it was constructed as follows:—A large mass of wood is carefully suspended so as to turn freely round a knife edge (Art. 142); the shot is fired into this mass, which is backed with iron plates to prevent the ball passing through or shivering it, so that the shot stays in it, and by the blow causes it to revolve through a certain angle (θ), the magnitude of which can be ascertained by a riband attached to a point of the pendulum which is pulled through a spring sufficiently strong to keep the riband straight while the mass moves up, and also to prevent any of it returning when the mass moves back; it is evident that the length of the riband gives the chord of the arc described by the point to which it is fastened, and thus θ is observed; the weight W of the pendulum includes that of the shot w; the distance h of the centre of gravity of W from the knife edge is determined in the manner suggested by Ex. 791. The radius of gyration is inferred from n, the number of small oscillations made in a minute; the distance, a, below the point of support of the point in which the shot strikes the pen-

dulum is measured; and it is (of course) endeavoured that this point should as nearly as possible coincide with the centre of percussion. From these data the velocity V of the shot can be found.

Ex. 793.—In the ballistic pendulum show that

$$V = \frac{120 g h W}{\pi n a w} \sin\frac{\theta}{2}.$$

APPENDIX I.

ON LIMITS, AND ON THE CYCLOID.

THROUGHOUT the present work particular geometrical limits have been used instead of the formulæ of the differential and integral calculus—at least this has been done as far as possible; if the reader has not been accustomed to reason on limits, he may perhaps find a difficulty in understanding the propositions in which they occur; should this be so the following remarks may prove useful:

1. *Definition of a limit.*—Let there be any variable magnitude X and let there be a fixed magnitude A; also suppose that X in the course of its successive changes continually approaches A, but never becomes equal to it, though the difference between the two magnitudes can be made less than any assigned magnitude however small; A is then said to be the limit of X. Thus suppose that X denotes the area of a polygon of n sides inscribed in a circle whose area is A; if we continually increase the number of sides, X will continually approach A; also if we assign any magnitude, say one square inch, a polygon with a certain number of sides can be found, whose area will differ from A by less than one square inch; in like manner if $\frac{1}{10}$th, $\frac{1}{100}$th, &c. of a square inch had been assigned; therefore the area of a circle is the limit of the area of the inscribed polygon.

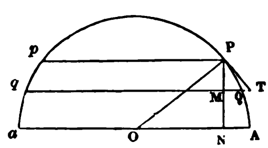
The simplest form which the reasoning on limits can assume is the following:—Suppose it can be proved that two variable quantities X and Y remain equal throughout their variations, and suppose that X continually approaches a limit

A, while Y approaches B, then it follows that A must equal B; this admits of a demonstration which we have not space to give. Thus it can be proved that the area of the inscribed regular polygon equals the rectangle between the semi-perimeter and the perpendicular let fall from the centre on one side; now the limit of the former is the area of the circle, and of the latter the rectangle between the semi-circumference and semi-diameter, and therefore the area of the circle *equals* that rectangle; not, the reader will observe, nearly equals it, but actually equals it. Prop. 1 supplies a good example of the same form of reasoning.

2. *On ultimate ratios.* — Suppose there are two variable magnitudes x and y whose separate limits are zero; what, it may be asked, is the limit of their ratio $\frac{x}{y}$? The value of this limit depends upon circumstances, and in different cases may have values differing to any extent whatever. Suppose x denotes the sine of an arc, and y the length of an arc, when x continually diminishes y continually diminishes, and their separate limits are zero; it is capable of proof that in this case the limit of $\frac{x}{y}$ is unity; but if x denotes the base and y the hypothenuse of a right-angled triangle, whose dimensions continually diminish in such a manner that angle (A) between x and y continues unchanged, then, although the separate limits of x and y are zero, the limit of $\frac{x}{y}$ is cos A; in the former case x is frequently said to be ultimately equal to y; in the latter x ultimately equals y cos A.

As this point is of great importance we will illustrate it by

Fig. 162.



the following case;—let APa be a semicircle, take P any point in its circumference, join P with the centre O, and draw

P N at right angles to A O; take Q a point between A and P, draw Q M q and P p parallel to A a; let P T be a tangent to the circle at P, and produce M Q to meet P T in T. Now if we suppose Q to move along the circumference up to P, then it is plain that the limiting values of P M, P Q, P T, M Q, M T, and Q T are separately zero, while P p is the limiting value of q M, q Q, and q T. Under these circumstances it is commonly stated that P M Q *is ultimately a triangle similar to* O P N; this means that the limit of $\frac{MQ}{PM}$ equals $\frac{PN}{ON}$, from whence it will of course follow that the limit of $\frac{PM}{PQ}$ equals $\frac{ON}{OP}$, and that of $\frac{MQ}{PQ}$ equals $\frac{PN}{OP}$. Now it will be remarked that $\frac{MT}{PM}$ equals $\frac{PN}{ON}$ under all circumstances; and therefore in the limit; so that what we have to prove will be done if we can show that the limit of $\frac{MQ}{PM}$ equals that of $\frac{MT}{PM}$, *i.e.* equals that of $\frac{MQ}{PM}+\frac{QT}{PM}$, or in other words we have to show that the limit of $\frac{QT}{PM}$ is zero. Now $QT \cdot Tq = PT^2$ (Eucl. 36—III.)

$$\therefore \frac{QT}{PM} = \frac{PT}{Tq} \cdot \frac{PT}{PM} = \frac{PT}{Tq} \cdot \text{Sec. } AOP.$$

Now the limit of P T is zero, while that of T q is P p, consequently in the limit the right hand side of this equation equals zero, and therefore the limit of $\frac{QT}{PM}=0$. The reader is requested to remark particularly that not only does Q T vanish in the limit, for so also does Q M and P Q, but that in the limit *it vanishes in comparison with them.** Hence if we are reasoning upon the relations that exist between the limits of the

* It is not unusual to call P T, P M, Q M, T M, small quantities of the first order; and Q T a small quantity of the second order. The reason of this is that Q T bears to a finite line the same ratio that the square of P T bears to the same or some other finite line.

ratios of the sides of P Q M we may substitute for them those of P T M, or *vice versâ*, the two being ultimately equal.

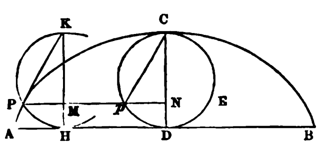
3. *Definition of the cycloid.* — The demonstrations that follow of the fundamental properties of the cycloid are very instructive examples of the mode of reasoning on ultimate values; they are inserted for that reason, and also because a knowledge of the properties is presupposed in Prop. 28.

Def. — If a circle rolls along a straight line, and keeps during the motion in one plane, a point in its circumference traces out a curve called a cycloid.

Let A B (in the figure to the following article) be the straight line, A C B the cycloid; take D, the middle point of A B, and draw C D at right angles to it; on C D describe a circle C D E; this is called the generating circle; take P, any point on the cycloid, draw P N at right angles to C D, cutting the circumference of the generating circle in *p*, join C *p*; then C *p* is parallel to the tangent at P, and the arc C P is twice as long as the chord C *p*. Also it is evident that A B must equal the circumference of the generating circle, since each point of the one has been successively in contact with each point of the other; for the same reason A H equals the length of the arc P H.

4. *To draw a tangent to a cycloid.*—Let the describing circle be in the position H P K, so that P is the describing point, then it is plain that the circle is at that instant turning on the point H; consequently P is moving in a direction at right angles to P H, that is P K is the tangent to the cycloid at the point P (Eucl. 31—III.). To show that P K is parallel to C *p*; since M D is a parallelogram, M H equals N D, therefore C N equals K M, and therefore P M equals *p* N (Eucl. 35—III.); therefore the right-angled triangles P M K and *p* N C are equal (Eucl. 4—I.) and the angle K P M equals the angle C *p* N, therefore C *p* is parallel to P K (Eucl. 27—I.).

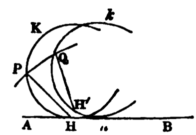
Fig. 163.



Remark.—There is a point in the above proof which requires

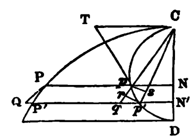
attention; let K P H be one position of the describing circle, and P the describing point; k Q h another position of the describing circle, then P H comes into the position Q H′. Why, it may be asked, are we entitled to neglect the distance H′ H, which we do in assuming that the point P ultimately moves into the next position Q by revolving round H? The answer to this is obvious enough when once the difficulty is clearly seen; the distance H H′ not only vanishes in the limit, for so also do P Q and H h, but it vanishes *in comparison with them.* (See Art. 2, App. I.) The same remark applies to the case in which A B is an arc of a circle, *i. e.* in which P is describing an epicycloid (Art. 96).

Fig. 164.



5. *To determine the length of the arc of a cycloid.* — Take P, any point in the arc of the cycloid; draw P N parallel to A D and cutting the generating circle in p; take P′ a point near P, and in like manner draw P′p'N′; join Cp and produce it to cut P′N′ in q; join Cp', draw ps and $p'r$ at right angles to Cp' and Cq respectively; draw C T, a tangent to the generating circle at C, join $p'p$ and produce it to cut that tangent in T; also draw P Q, a tangent to the cycloid at P.

Fig. 165.

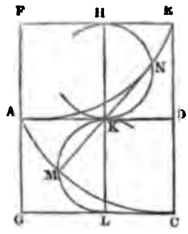


Since Pq is a parallelogram, pq equals P Q, *i. e.* it ultimately equals P P′ the increment of the arc of the cycloid; again, since Cp ultimately equals Cs (for ultimately the angle p C s vanishes and the triangle becomes isosceles), sp' ultimately equals the increment of the chord. Now the chord Tpp' is ultimately a tangent to the circle at p, therefore ultimately the angle TpC equals T Cp, *i. e.* equals pqp' (Eucl. 29—I.); therefore ultimately $pp'q$ is an isosceles triangle, and therefore ultimately pq is the double of pr. But since C ps is ultimately a right angle, spr is also ultimately a right angle, and the figure $prp's$ ultimately rectangular, and pr ultimately equal to sp'. Therefore P P′ is ultimately double of sp'; or the increment of

the arc is ultimately double of the increment of the chord. Now both the chord and the arc commence from the same point C, but when two magnitudes both commence from zero, and the first *grows* twice as fast the second, any value of the first will be twice as large as the corresponding value of the second; therefore the length of the arc C P is double that of the corresponding chord C *p*.*

6. *To cause a point suspended by a perfectly flexible string to oscillate in a cycloid.*—Let A C D be half the cycloid in which the point is to be made to vibrate, the vertex of that cycloid being C; take an equal semicycloid A E F, and place it with its vertex at A, the bases of the two cycloids being parallel, complete the rectangle G E. Now if the point be suspended from E, and the length of the string equals A E, then if the string be wrapped on A E, the point in the process of unwrapping the string will describe A C; and of course in the same manner it can be made to describe the other half of A C. This can be proved as follows :—

Fig. 166.



Draw any line H K L at right angles to F E and A D, and the circles H N K and K M L; if we suppose these to be the describing circles of the respective cycloids, N will be the describing point of E A, the circle having rolled from E to H, and M the describing point of A C, the circle having rolled from A to K; join M K and K N. Now F E equals the semicircumference of H N K or K M L, H N equals H E, and F H equals A K equals the arc M K; therefore M L equals H E, *i.e.* H N, therefore the angle H K N equals the angle M K L (Eucl. 27—III.), and therefore M K and K N are in the same straight line. Also since the arcs M K and K N are equal, the

* The student will find the above proposition a very instructive one; he must bear in mind that the reasoning has reference not to the lines in the diagrams, but to what they tend to become when P P′ is continually diminished.

chords are also equal; therefore M N, which touches the cycloid in N, equals twice K N, *i. e.* equals the arc A N. But if the thread were unwrapped it would equal A N in length, and be a tangent at N, *i.e.* it would coincide with N M; and this being true for all positions of N, the heavy point will describe the cycloid A C.

APPENDIX II.

ON MOTION IN A RESISTING MEDIUM.

It has been already stated that the motion of a military projectile differs largely from the motion assigned by the parabolic theory, and that this difference arises from the resistance of the air which it experiences in fact, and which is neglected in that theory. The following pages contain one or two simple cases of the motion of a projectile, in which the resistance of the air is taken into account: the reader must, however, be on his guard against supposing that these investigations completely represent the facts of the motion of military projectiles.*

The resistance which the air offers to the motion of a spherical shot may be nearly expressed by the following formula†:—

$$R = 0{\cdot}0006\, A v^2 \left(1 + \frac{v}{1300}\right)$$

where R is the resistance in lbs., A the area of a great circle of the shot in square feet, and v the velocity of the shot in feet per second. It will be observed that if v

* The deficiency alluded to in the text is twofold; (1) as is stated in the next paragraph the resistance is not accurately represented by kv^2; (2) no account is given of the *deviation* of the shot, which depends on the form of the shot and on the axis about which it rotates. This deviation, especially in the case of spherical shot, is frequently very great. The reader will find an account of some very beautiful experiments illustrative of the deviation of shot in a paper by M. Magnus translated in Taylor's Scientific Memoirs.

† Poncelet, Introd. à la Mécan. Indust. p. 618—621.

is small, $1 + \frac{v}{1300}$ very nearly equals unity, and also that for any body whose velocity does not change greatly the *variations* of the term $1 + \frac{v}{1300}$ are very small, and hence, in both these cases, the resistance R can be denoted by the formula kv^2; in what follows this will be assumed to be the law according to which the resistance changes.

A body acted on only by the resistance of the air begins to move with a given velocity, to determine its motion.

Let $\frac{W}{g}$ be the mass of the body, and V be its initial velocity, z the space described, and v its velocity at the end of t seconds; z', v', the space and velocity at the end of t' seconds. Now there were $\frac{W}{2g}v^2$ units of work accumulated in it at the end of t seconds, and $\frac{W}{2g}v'^2$ at the end of t' seconds; but ultimately the resistance of the medium may be considered constant throughout the short space $z'-z$, and to equal kv^2; therefore while the body describes that short space it will do $kv^2(z'-z)$ units of work. Hence,

$$\frac{W}{2g}v^2 = \frac{W}{2g}v'^2 + kv^2(z'-z) \text{ ultimately}$$

$$\text{or } \frac{W}{2g} \cdot \frac{dv^2}{dz} = -kv^2$$

$$\therefore \frac{dv^2}{dz} = -2\beta v^2$$

where β represents $\frac{gk}{W}$.

$$\therefore v^2 = Ce^{-2\beta z}$$

Now, when $z = 0$, $v = V$, $\therefore C = V^2$, and we obtain

$$v^2 = V^2 e^{-2\beta z}$$

$$\text{or } v = Ve^{-\beta z} \tag{1}$$

which gives the velocity which the body has after z feet have been described ; it will be remarked that as z increases v continually diminishes, but does not vanish for any finite value of z, so that the resistance of the air only would never entirely stop the body.

Next to determine the space z described in t seconds ; the space $z'-z$ is ultimately described in the time $t'-t$, with a uniform velocity v; hence,

$$z'-z = v\,(t'-t) \text{ ultimately,}$$

$$\text{or, } \frac{dz}{dt} = v = \mathrm{V}e^{-\beta z}$$

$$\therefore\ \beta \mathrm{V} t = e^{\beta z} + \mathrm{C}$$

Now, when $t = 0$, $z = 0$ $\therefore$ $\mathrm{C} = -1$, or

$$e^{\beta z} = \beta \mathrm{V} t + 1$$

$$\therefore\ z = \frac{1}{\beta} \log\,(\beta \mathrm{V} t + 1) \qquad (2)$$

From (1) and (2) we immediately obtain

$$\frac{1}{v} - \frac{1}{\mathrm{V}} = \beta t. \qquad (3)$$

which gives the relation between the velocity and the time. The equations (1) (2) (3) completely determine the motion.

If a body moves in a vertical line in a resisting medium and under the action of gravity, to determine the motion.

There are three cases included under this general enunciation; (1) if the body is thrown upward with a given velocity; (2) if it is thrown downward with a given velocity; (3) if it is dropped. The three cases closely resemble each other and their solution is step by step the same as that in the last article ; we will therefore indicate the principal steps in the first case and mention the results in the two others.

(1.) The fundamental equation is the following—where the notation is the same as in the last article

$$\frac{\mathrm{W}}{2g} v^2 = \frac{\mathrm{W}}{2g} v'^2 + (\mathrm{W} + k v^2)\,(z' - z) \text{ ult.}$$

$$\text{or } -\frac{W}{2g}\cdot\frac{dv^2}{dz} = W + kv^2$$

$$\therefore\ v^2 = a^2\left\{(1+\gamma^2)\,e^{-2\beta z} - 1\right\} \qquad (4)$$

where $a^2 = \frac{W}{k}$ and $\gamma^2 = V^2\frac{k}{W}$.

$$\text{Hence } \frac{dz}{dt} = a\sqrt{(1+\gamma^2)\,e^{-2\beta z} - 1}.$$

$$\therefore\ a\beta t = \sin^{-1}\frac{y}{\sqrt{1+\gamma^2}} + C$$

$$= C_1 - \cos^{-1}\frac{y}{\sqrt{1+\gamma^2}}.$$

where $y = e^{\beta z}$; but if $t = 0$, $z = 0$, and $y = 1$.

$$\therefore\ a\beta t = \sin^{-1}\frac{y}{\sqrt{1+\gamma^2}} - \sin^{-1}\frac{1}{\sqrt{1+\gamma^2}}$$

$$= \sin^{-1}\frac{\gamma y - \sqrt{1+\gamma^2-y^2}}{1+\gamma^2}$$

$$\text{and } a\beta t = -\cos^{-1}\frac{y}{\sqrt{1+\gamma^2}} + \cos^{-1}\frac{1}{\sqrt{1+\gamma^2}}$$

$$= \cos^{-1}\frac{y + \gamma\sqrt{1+\gamma^2-y^2}}{1+\gamma^2}$$

$$\therefore\ y \text{ or } e^{\beta z} = \gamma\sin a\beta t + \cos a\beta t. \qquad (5)$$

(2.) In the second case the fundamental equation is

$$\frac{W}{2g}v'^2 = \frac{W}{zg}v^2 + (W - kv^2)(z' - z) \text{ ult.}$$

$$\text{or } \frac{W}{2g}\frac{dv^2}{dz} = W - kv^2$$

and we obtain

$$v^2 = a^2\left\{1 - (1-\gamma^2)\,e^{-2\beta z}\right\} \qquad (6)$$

$$e^{\beta z} = \tfrac{1}{2}\left\{(1+\gamma)\,e^{-a\beta t} + (1-\gamma)\,e^{a\beta t}\right\} \qquad (7)$$

It will be observed in equation (6) that as z increases v continually approaches a, and can never exceed it; a or $\sqrt{\frac{W}{k}}$ is therefore called the ultimate velocity and is soon sensibly reached by a falling body.

(3.) The third case differs from the second in that $V = 0$; *i.e.* $\gamma = 0$

$$\text{or } v^2 = a^2 \left(1 - e^{-2\beta z}\right) \tag{8}$$

$$e^{\beta z} = \tfrac{1}{2}\left\{e^{-a\beta t} + e^{a\beta t}\right\} \tag{9}$$

*To determine approximately the motion of a projectile thrown at a small elevation, taking into account the resistance of the air.**

Let $x\,y$ be the coordinates of the body's position at the end of t seconds; let z be the space which the body would have described along its direction of projection if not acted on by gravity; α the inclination to the horizon of the direction of projection; then, as in Art. (109), we shall have

$$x = z \cos \alpha$$
$$y = z \sin \alpha - \tfrac{1}{2} g t^2$$

approximately; the value of y, it will be observed, supposes that the motion of the shot, so far as it depends on gravity, is unaffected by the resistance of the air. But from equation (2)

$$e^{\beta z} = 1 + \beta V t$$

$$\therefore\ t = \frac{1}{\beta V}\left\{e^{\beta x \sec \alpha} - 1\right\}$$

$$\text{or } y = x \tan \alpha - \frac{g}{2\beta^2 V^2}\left\{e^{\beta x \sec \alpha} - 1\right\}^2 \tag{10}$$

which is an approximate expression for the path of a projectile. Differentiating we obtain

$$\frac{dy}{dx} = \tan \alpha - \frac{g \sec \alpha}{\beta V^2}\left\{e^{\beta x \sec \alpha} - 1\right\} e^{\beta x \sec \alpha} \tag{11}$$

Now let x_1 be the range on the horizontal plane, and x_2 the value of x corresponding to the greatest height of the projectile, we obtain from (10) and (11)

$$0 = x_1 \tan \alpha - \frac{g}{2\beta^2 V^2}\left\{e^{\beta x_1 \sec \alpha} - 1\right\}^2 \tag{12}$$

$$0 = \tan \alpha - \frac{g \sec \alpha}{\beta V^2}\left\{e^{\beta x_2 \sec \alpha} - 1\right\} e^{\beta x_2 \sec \alpha} \tag{13}$$

* This Article is in substance due to the Rev. H. Moseley.

Finally to determine the velocity of the shot at any point. We have

$$v^2=\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2=\left(\frac{dz}{dt}\right)^2\cos^2\alpha+\left(\frac{dz}{dt}\sin\alpha-gt\right)^2$$

$$\text{or } v^2=\left(\frac{dz}{dt}\right)^2-2gt\frac{dz}{dt}\sin\alpha+g^2t^2$$

$$=\mathrm{V}^2 e^{-2\beta z}-\frac{2g}{\beta\mathrm{V}}(e^{\beta z}-1)\mathrm{V}e^{-\beta z}\sin\alpha+\frac{g^2}{\beta^2\mathrm{V}^2}(e^{\beta z}-1)^2$$

$$\therefore v^2=\mathrm{V}^2 e^{-2\beta x\sec\alpha}-\frac{2g}{\beta}(1-e^{-\beta x\sec\alpha})\sin\alpha$$

$$+\frac{g^2}{\beta^4\mathrm{V}^2}(e^{\beta x\sec\alpha}-1)^2 \qquad (14)$$

In the case of point blank firing we have $\alpha = 0$, and therefore, measuring y *downward*, we have from equation (10)

$$y=\frac{g}{2\beta^2\mathrm{V}^2}(e^{\beta x}-1)^2$$

$$\therefore\quad x=\frac{1}{\beta}\log\left(1+\beta\mathrm{V}\sqrt{\frac{2y}{g}}\right) \qquad (15)$$

which gives the point blank range. Also we have

$$\mathrm{V}^2=\frac{g}{2\beta^2 y}(e^{\beta x}-1)^2 \qquad (16)$$

which gives the velocity at point blank firing to hit a point (x, y). Lastly from equation (14) we obtain

$$v^2=\mathrm{V}^2 e^{-2\beta x}+2gy \qquad (17)$$

which gives the velocity with which it strikes the point (x, y).

*To determine the velocity of a bolt at any point of the rifled barrel out of which it is fired.**

The principle on which this is determined is as follows :—When the bolt is at any point of the barrel the work that has been done by the ignited powder will equal the work expended on the friction of the barrel and screw, together with the work expended on the resistance of the air, and the work accumu-

* Communicated by the Rev. H. Moseley.

lated in the bolt. In applying this principle we shall use the following notation. Let W denote the weight of the bolt, v its velocity at a distance x from the bottom of the chamber, ω its angular velocity at the same point, a the length of the chamber, r the radius of the barrel or bolt, P the friction due to the binding of the bolt on the surface of the barrel, being the pressure which would be necessary to drive the bolt forward if the barrel were not rifled, ϕ the limiting angle of resistance between the thread of the screw and the bolt, ι the inclination of the thread to the base of the screw, n the number of turns of the thread of the screw per foot of the length of the barrel, m the pressure of the air per square foot of the section of the bolt, being that which it would sustain if it were at rest and a vacuum were behind it, kv^2 the resistance of the air per square foot of the section of the bolt, E the elasticity per square foot of the gases when first liberated, I the moment of inertia of the bolt about its axis.

(1.) To find the work done by the gases. If we suppose Boyle's law to obtain in the expansion of the liberated gases, the work done by them on the bolt when it reaches the point x is given by the expression

$$\pi r^2 \int_a^x \frac{a\mathrm{E}}{x}\, dx = \pi \mathrm{E} a r^2 \log \frac{x}{a}.$$

(2.) To find the work expended on the friction of the barrel and screw. Let p be the pressure which acting parallel to the axis would just drive the bolt forward; then $p\,(x - a)$ is the number of units of work required; now p overcomes the reaction of the screw inclined to the normal at an angle ϕ, and also the resistance P which will act in a direction opposite to the motion, *i.e.* along the thread; hence from the properties of the inclined plane we obtain

$$\frac{p}{\mathrm{P}} = \frac{\cos\phi}{\sin(\iota - \phi)}$$

$$\therefore\ p\,(x - a) = \frac{\mathrm{P}\,(x - a)\cos\phi}{\sin(\iota - \phi)}.$$

(3.) To find the work expended on the resistance of the air. This results from the pressure the bolt would have to sustain

if it were at rest and a vacuum were behind it, and from the resistance which the air opposes to the motion of the bolt. The former is represented by the expression $\pi r^2 m (x - a)$ and the latter by $\pi r^2 k \int_a^x v^2 \, dx$.

(4.) To find the work accumulated in the bolt. This will equal $\frac{1}{2} \mathrm{I} \omega^2 + \frac{\mathrm{W}}{2g} v^2$.

Now from the form of the screw

$$2 \pi r n \tan \iota = 1.$$

But the bolt makes n revolutions in traversing one foot, and therefore $v n$ in traversing v feet, and therefore its angular velocity is $2 \pi v n$

$$\therefore \omega = \frac{v}{r \tan \iota}$$

and the work accumulated in the bolt will equal

$$\frac{v^2}{2} \left\{ \frac{\mathrm{I}}{r^2 \tan^2 \iota} + \frac{\mathrm{W}}{g} \right\}.$$

(5.) Hence if we collect all these results we obtain for the determination of v the equation

$$\pi \mathrm{E} a r^2 \log \frac{x}{a} = \frac{\mathrm{P} (x - a) \cos \phi}{\sin (\iota - \phi)} + \pi m r^2 (x - a)$$
$$+ \pi k r^2 \int_a^x v^2 \, dx + \left(\frac{\mathrm{I}}{r^2 \tan^2 \iota} + \frac{\mathrm{W}}{g} \right) \frac{v^2}{2}$$

an equation which, by obvious abbreviations, may be written in the form

$$v^2 + \mathrm{A} \int_a^x v^2 \, dx = \mathrm{C} \log \frac{x}{a} - \mathrm{B} (x - a);$$

in which A is always a small quantity since k enters it as a factor. Differentiating we obtain

$$\frac{d v^2}{d x} + \mathrm{A} v^2 = \frac{\mathrm{C}}{x} - \mathrm{B}. \qquad (18)$$

An equation which is easily integrated, and gives

$$v^2 e^{Ax} = C\int_a^x \frac{e^{Ax}}{x}\, dx - \frac{B}{A}(e^{Ax} - e^{Aa})$$

which is the result required; it being borne in mind that

$$\int \frac{e^{Ax}}{x}\, dx = \log x + Ax + \frac{A^2 x^2}{1 \,.\, 2^2} + \frac{A^3 x^3}{1 \,.\, 2 \,.\, 3^2} + \ldots .$$

(6.) A first and second approximation may be obtained as follows: — In equation (18) assume $A = 0$

$$\therefore \; v^2 = C \log \frac{x}{a} - B\,(x - a)$$

which is the first approximation. Next substitute this value of v^2 in the small term of equation (18) and we obtain

$$\frac{dv^2}{dx} = \frac{C}{x} - B - A\left\{C \log \frac{x}{a} - B\,(x - a)\right\}.$$

Integrate between the limits of x and a, and we obtain

$$v^2 = C \log \frac{x}{a} - B\,(x - a) - A\left\{C\, x \log \frac{x}{a} \right.$$

$$\left. - C\,(x - a) - \tfrac{1}{2}\, B\,(x - a)^2\right\}. \qquad (19)$$

(7.) Lastly, if we wish to determine at what point v is a maximum, we have $\frac{dv^2}{dx} = 0$, and therefore approximately

$$0 = \frac{C}{x} - B$$

$$\text{or } x = \frac{C}{B}. \qquad (20)$$

a value which renders $\frac{d^2 v^2}{a x^2}$ *negative*, and therefore v^2 a maximum. The value of x above determined is an approximate

expression for the length of the gun in order that, *cæteris paribus*, the shot may leave it with the greatest possible velocity; if for B and C we write the expressions for which they stand, the equation (20) takes the form

$$x = \frac{\pi E a r^2 \sin(\iota - \phi)}{P \cos\phi + \pi m r^2 \sin(\iota - \phi)}. \tag{21}$$

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